ELLIPTIC QUANTUM GROUPS AND THEIR FINITE–DIMENSIONAL REPRESENTATIONS

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, τ a point in the upper half–plane, and $\hbar \in \mathbb{C}$ a deformation parameter such that the image of $\hbar$ in the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is of infinite order. In this paper, we give an intrinsic definition of the category of finite–dimensional representations of the elliptic quantum group $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ associated to $\mathfrak{g}$. The definition is given in terms of Drinfeld half–currents and extends that given by Enriquez–Felder for $\mathfrak{g} = \mathfrak{sl}_2$ [14]. When $\mathfrak{g} = \mathfrak{sl}_n$, it reproduces Felder's RLL definition [23] via the Gauß decomposition obtained in [14] for $n = 2$ and [28] for $n \geq 3$. We classify the irreducible representations of $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ in terms of elliptic Drinfeld polynomials, in close analogy to the case of the Yangian $Y_\hbar(\mathfrak{g})$ and quantum loop algebra $U_q(L\mathfrak{g})$ of $\mathfrak{g}$. An important ingredient in the classification, which circumvents the fact that $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ does not appear to admit Verma modules, is a functor from finite–dimensional representations of $U_q(L\mathfrak{g})$ to those of $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, which is an elliptic analogue of the monodromy functor constructed in [30]. Our classification is new even for $\mathfrak{g} = \mathfrak{sl}_2$, and holds more generally when $\mathfrak{g}$ is a symmetrisable Kac–Moody algebra, provided finite–dimensionality is replaced by an integrability and category $\mathcal{O}$ condition.

Contents

1. Introduction

1.1. Let $\mathfrak{g}$ be a finite–dimensional simple Lie algebra over $\mathbb{C}$. The aim of this paper is to define and study the category of finite–dimensional representations of the elliptic quantum group associated to $\mathfrak{g}$. The results of this paper hold more generally for any symmetrisable Kac–Moody algebra, provided finite–dimensionality is replaced by an integrability and category $\mathcal{O}$ condition, and the main body of

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the paper deals with this generality. To simplify the exposition, we restrict our attention to $\mathfrak{g}$ of finite type in the introduction.

1.2. Since Baxter's solution of the eight–vertex model [3, 4], and Felder's introduction of the dynamical Yang–Baxter equations [23], there have been numerous proposals to define elliptic quantum groups (see Section 1.10 below for a more detailed history of the subject). One of the contributions of this paper is such a definition which is intrinsic, uniform for all Lie types, and valid for numerical values of the deformation and elliptic parameters.

We fix throughout a complex number τ in the upper half plane $\mathbb{H}$, and a deformation parameter $\hbar \in \mathbb{C}$ such that the image of $\hbar$ in the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is of infinite order. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra, and $\mathbf{I}$ the set of vertices of the Dynkin diagram of $\mathfrak{g}$ relative to a choice of simple roots in $\mathfrak{h}^*$. In Section 2, we define the category $\mathrm{Rep}_{\mathrm{fd}}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$ of finite–dimensional representations of the elliptic quantum group associated to $\mathfrak{g}$. An object of this category is a finite–dimensional semisimple $\mathfrak{h}$–module $\mathbb{V}$, together with a collection of meromorphic, $\mathrm{End}(\mathbb{V})$–valued functions $\{\Phi_i(u,\lambda), \mathfrak{X}_i^\pm(u,\lambda)\}_{i\in\mathbf{I}}$ of a spectral variable $u \in \mathbb{C}$ and dynamical parameter $\lambda \in \mathfrak{h}^*$

This data is subject to certain periodicity conditions and prescribed commutation relations, which are elliptic analogues of the relations satisfied by the half–currents in Drinfeld's new realization of Yangians and quantum loop algebras. Our definition extends that given by Enriquez–Felder for $\mathfrak{g} = \mathfrak{sl}_2$ [14] and, for $\mathfrak{g} = \mathfrak{sl}_n$, reproduces Felder's RLL category [23] via the Gauß decomposition obtained in [14] for $n = 2$, and in [28] for $n \geq 3$.[1]

1.3. We restrict our attention to the full subcategory $\mathcal{L}_{\hbar,\tau}(\mathfrak{g}) \subset \mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$ consisting of those objects for which the endomorphisms $\Phi_i(u,\lambda)$ are independent of the dynamical parameter λ. The main result (Theorem 9.1) of this paper is a classification of the irreducible objects in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ in terms of elliptic analogues of Drinfeld polynomials. Let for this purpose

$$SE_\tau = \bigcup_{n\geq 0} E_\tau^n/\mathfrak{S}_n$$

be the union of the symmetric powers of the elliptic curve E_τ. A point in SE_τ may be thought of as encoding the zeros of an elliptic polynomial.

Theorem. *The set of isomorphism classes of irreducible objects in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is in bijection with $|\mathbf{I}|$ copies of SE_τ*

$$\mathrm{Irr}(\mathcal{L}_{\hbar,\tau}(\mathfrak{g})) \xrightarrow{\sim} \underbrace{SE_\tau \times \cdots \times SE_\tau}_{|\mathbf{I}|} \tag{1.1}$$

1.4. Theorem 1.3 is formally analogous to the classification of finite–dimensional irreducible representations of Yangians and quantum loop algebra by Drinfeld polynomials [9, 13]. Namely, given an $\mathbf{I}$–tuple $\mathbf{b}_i = \{b_i^{(j)}\}_{j=1}^{N_i} \in \mathbb{C}^{N_i}/\mathfrak{S}_{N_i}$, $i \in \mathbf{I}$, there is an irreducible object $\mathbb{V}(\mathbf{b})$ in the category $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$, which contains a non–zero vector Ω such that

- $\mathfrak{X}_i^+(u,\lambda)\,\Omega = 0$ for any $i \in \mathbf{I}$, $u \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*$.

[1] A closely related decomposition, valid when both $\hbar$ and τ are formal parameters, was recently obtained by Konno [36].

- For any $i \in \mathbf{I}$,
$$\Phi_i(u)\,\Omega = \prod_{j=1}^{N_i} \frac{\theta(u - b_i^{(j)} + d_i\hbar)}{\theta(u - b_i^{(j)})}\,\Omega \tag{1.2}$$
where $\theta(x)$ is the odd theta function (see Section 2.2) and $\{d_i\}$ are the symmetrising integers for the Cartan matrix of $\mathfrak{g}$.
- $\mathbb{V}$ is spanned by vectors obtained by successively applying the lowering operators $\mathfrak{X}_i^-(u,\lambda)$ to Ω.

Moreover, shifting a point $b_i^{(j)}$ by an element of lattice $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ yields an isomorphic representation, and this condition is necessary and sufficient for two irreducible objects $\mathbb{V}(\mathbf{b})$ and $\mathbb{V}(\mathbf{c})$ to be isomorphic.

1.5. The proof of Theorem 1.3 can be broken down into the following steps

1. A triangularity result (Theorem 3.8), according to which an irreducible object $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ contains a (unique up to scalar) non–zero weight vector Ω which is annihilated by the raising operators, is an eigenvector for $\{\Phi_i(u)\}_{i\in\mathbf{I}}$, and is such that $\mathbb{V}$ is spanned by successively applying lowering operators to Ω.
2. Theorem 4.1 states that the eigenvalues of $\{\Phi_i(u)\}$ on Ω are of the form (1.2).

These two steps give rise to a well–defined classifying map (1.1). For Yangians and quantum loop algebras, the injectivity of the analogous map is proved using Verma modules. For elliptic quantum groups, Verma modules do not seem to exist, however. We circumvent this obstacle as follows.

3. Let $U_q(L\mathfrak{g})$ be the quantum loop algebra of $\mathfrak{g}$, where $q = e^{\pi\iota\hbar}$. In Section 6, we extend the construction of [29, 30] to the elliptic setting, and obtain a monodromy functor
$$\Theta : \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \longrightarrow \mathrm{Rep}_{\mathrm{fd}}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$$
whose image lies in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. We further prove that Θ is essentially surjective, and maps irreducibles to irreducibles (Theorem 8.6). These results allow us to rely on the classification of finite–dimensional irreducible representations of $U_q(L\mathfrak{g})$ [13, 9], and prove that the map (1.1) is in fact a bijection (Theorem 9.1). We sketch the construction of the functor Θ below.

1.6. Let $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ be the category of finite–dimensional representations of the quantum loop algebra $U_q(L\mathfrak{g})$. By using half–currents, an object of $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ may be viewed as a finite–dimensional vector space $\mathcal{V}$, together with a collection of rational, $\mathrm{End}(\mathcal{V})$–valued functions $\{\Psi_i(z), \mathcal{X}_i^\pm(z)\}_{i\in\mathbf{I}}$ of a complex variable z, which are regular at $z = 0, \infty$ and satisfy certain commutation relations (see Section 5.2).

Let $\mathcal{V} \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$. The action of the elliptic quantum group on $\Theta(\mathcal{V}) = \mathcal{V}$ is obtained as follows.[2]

[2]Strictly speaking, the functor Θ is only defined on a dense subcategory of $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, but we will gloss over this point in the introduction.

- Set $p = e^{2\pi\iota\tau}$, so that $|p| < 1$. Let $K_i = \Psi_i(\infty) = \Psi_i(0)^{-1} \in GL(\mathcal{V})$, and define
$$G_i^{\pm}(z) = \prod_{n\geq 1} K_i^{\pm 1}\Psi_i(p^{\pm n}z)$$
The $GL(\mathcal{V})$–valued function $G_i^+(z)$ and $G_i^-(z)$ are meromorphic on $\mathbb{C}^\times$, holomorphic in a neighbourhood of $z = 0, \infty$ respectively, and such that $G_i^+(0) = 1 = G_i^-(\infty)$. By construction, $G_i^{\pm}(z)$ are the canonical fundamental solutions near $z = 0, \infty$ of the p–difference equations
$$G_i^+(pz) = \left[K_i\Psi_i(pz)\right]^{-1} G_i^+(z) \qquad G_i^-(pz) = \left[K_i^{-1}\Psi_i(z)\right] G_i^-(z)$$
determined by the commuting field $\Psi_i(z)$ of $U_q(L\mathfrak{g})$.
- The action of the commuting field $\Phi_i(u)$ of $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ on $\Theta(\mathcal{V})$ is given by
$$\Phi_i(u) = G_i^+(z)\Psi_i(z)G_i^-(z)\big|_{z=e^{2\pi\iota u}} \tag{1.3}$$
By construction, $\Phi_i(u)$ is a doubly quasi–periodic function of u, that is satifies
$$\Phi_i(u+1) = \Phi_i(u) \qquad \text{and} \qquad \Phi_i(u+\tau) = K_i^{-2}\Phi_i(u)$$
and is the monodromy of the p–difference equation defined by $\Psi_i(z)$.
- The raising and lowering operators act on $\Theta(\mathcal{V})$ by
$$\mathfrak{X}_i^{\pm}(u,\lambda) = \frac{\theta^+(0)}{\theta^+(d_i\hbar)} \int_{C_i^{\pm}} \frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)} G_i^{\pm}\left(e^{2\pi\iota v}\right) \mathcal{X}_i^{\pm}\left(e^{2\pi\iota v}\right) dv$$
where $\theta^+(x) = \prod_{n\geq 1}(1-p^n e^{2\pi\iota n x})$ and $\lambda_i = (\lambda, \alpha_i)$. The choice of contours $C_i^{\pm}$ is explained in Section 6.2.

Theorem. *The above contruction gives rise to an exact, faithful functor*
$$\Theta : \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \longrightarrow \mathcal{L}_{\hbar,\tau}(\mathfrak{g}) \subset \mathrm{Rep}_{\mathrm{fd}}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$$

1.7. We now state a number of additional properties of the functor Θ, which mirror those of the functor constructed in [30] and are used in the proof of Theorem 1.3.

The action of Θ on highest weights is easily determined. Let $\mathcal{V} \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ be irreducible, with Drinfeld polynomials $\{P_i(w)\}_{i\in\mathbf{I}}$
$$P_i(w) = \prod_{k=1}^{N_i}(w - \beta_k^{(i)})$$
where $\beta_k^{(i)} \in \mathbb{C}^\times$ for each $i \in \mathbf{I}$ and $1 \leq k \leq N_i$. Thus, if $\Omega \in \mathcal{V}$ is the (unique up to scaling) highest–weight vector, we have
$$\Psi_i(z)\,\Omega = \prod_{k=1}^{N_i} \frac{q_i z - q_i^{-1}\beta_k^{(i)}}{z - \beta_k^{(i)}}\,\Omega$$
where $q_i = q^{d_i}$. By (1.3), and Jacobi's triple product identity
$$\theta(u) = \frac{\theta^+(u)\sin(\pi u)\theta^-(u)}{\theta^+(0)^2}$$
where $\theta^{\pm}(u) = \prod_{n\geq 1}(1 - p^n e^{\pm 2\pi\iota n u})$, we get
$$\Phi_i(u)\,\Omega = \prod_k \frac{\theta(u - b_k^{(i)} + d_i\hbar)}{\theta(u - b_k^{(i)})}\,\Omega$$

where $b_k^{(i)} \in \mathbb{C}$ are such that $e^{2\pi\iota b_k^{(i)}} = \beta_k^{(i)}$. Thus, Θ maps the roots of Drinfeld polynomials for $U_q(L\mathfrak{g})$ to their image in E_τ.

1.8. Choose now a subset $\mathrm{S} \subset \mathbb{C}$ which is a fundamental domain for the action of $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ on $\mathbb{C}$, and is stable under shifts by $\mathbb{Z}\frac{\hbar}{2}$. Define the full subcategories

$$\mathrm{Rep}^{\mathrm{S}}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \subset \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \qquad \text{and} \qquad \mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g}) \subset \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$$

as follows. $\mathrm{Rep}^{\mathrm{S}}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ consists of those representations $\mathcal{V}$ for which the poles of $\{\Psi_i(z)^{\pm 1}\}$ lie in $e^{2\pi\iota\,\mathrm{S}}$. $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ consists of representations $\mathbb{V}$ for which the generalized eigenvalues of $\{\Phi_i(u)\}$ are of the following form

$$A_i(u) = \prod_k \frac{\theta(u - c_{i,k} + d_i\hbar)}{\theta(u - c_{i,k})} \prod_l \frac{\theta(u - c'_{i,l} - d_i\hbar)}{\theta(u - c'_{i,l})} \tag{1.4}$$

where $c_{i,k}, c'_{i,l} \in \mathrm{S}$.

Theorem.

(1) *The functor Θ restricts to $\Theta^{\mathrm{S}} : \mathrm{Rep}^{\mathrm{S}}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \to \mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$. This restriction sets up a bijection between simple objects of the two categories.*

(2) *Every object of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is isomorphic to an object of $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$.*

For a given $\mathbb{V} \in \mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$, we explicitly construct a representation $\mathcal{V} \in \mathrm{Rep}^{\mathrm{S}}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ such that $\mathbb{V} = \Theta(\mathcal{V})$ (see Theorem 8.6). This inverse contruction is analogous to [30, §6], and is achieved by solving a Riemann–Hilbert factorization problem for doubly–quasi periodic (abelian, matrix–valued) functions in Section 7. Combining Theorem 1.8 with the calculation given in Section 1.7 and the classification theorem of finite–dimensional irreducible representations of $U_q(L\mathfrak{g})$ proves Theorem 1.3.

1.9. One difference between Theorem 1.8 and the analogous relation between finite–dimensional representations of Yangians and quantum loop algebras obtained in [30] is that the functor Θ^{S} is not an equivalence. This is because the two categories are defined over different fields. Indeed, $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ is defined over $\mathbb{C}$, whereas the field of definition of $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ is

$$\mathrm{End}_{\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})}(\mathbf{1}) = \{\varphi : \mathfrak{h}^* \to \mathbb{C} \text{ meromorphic such that } \varphi(\lambda + \hbar\alpha_i) = \varphi(\lambda)\}$$

However, if $\mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ is the category with the same objects as $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$, and morphisms the $\mathbb{C}$–linear homomorphisms commuting with the operators $\{\Phi_i(u), \mathfrak{X}_i^{\pm}(u,\lambda)\}$, the following is an immediate corollary of the constructions leading to a proof of Theorem 1.8.

Corollary. *The functor $\Theta^{\mathrm{S}} : \mathrm{Rep}^{\mathrm{S}}_{\mathrm{fd}}(U_q(L\mathfrak{g})) \to \mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ is an isomorphism of categories.*

1.10. **Elliptic R–matrices.** To put the results of this paper in perspective, we now give some background on elliptic quantum groups. Their origins can be traced back to Baxter's elliptic R–matrix, which records the Boltzmann weights of the eight–vertex model [3, 4], and its generalisation $R_B(u)$ to higher rank by Belavin [6]. The study of the quantum groups corresponding to $R_B(u)$ was initiated by Sklyanin [42, 43] for the $n = 2$ case, and generalized by Cherednik [12] for arbitrary n. The finite–dimensional representations of these *Sklyanin algebras*, which may be thought of as elliptic deformations of the loop algebra of $\mathfrak{sl}_n$, and are also known as vertex–type elliptic algebras, are also investigated in these works.

1.11. **Dynamical Yang–Baxter equations.** It is natural to look for elliptic solutions of the Yang–Baxter equations (YBE) corresponding to any simple Lie algebra $\mathfrak{g}$. Belavin and Drinfeld proved that, for the classical YBE, such solutions only exist for $\mathfrak{g} = \mathfrak{sl}_n$ [7]. Partly to remedy this, Felder [23] introduced the *dynamical quantum Yang–Baxter equation* (DQYBE)

$$R_{12}(u, \lambda - \hbar \mathsf{h}^{(3)}) R_{13}(u+v, \lambda) R_{23}(v, \lambda - \hbar \mathsf{h}^{(1)}) = R_{23}(v, \lambda) R_{13}(u+v, \lambda - \hbar \mathsf{h}^{(2)}) R_{12}(u, \lambda) \quad (1.5)$$

where $\mathfrak{h}$ is an abelian Lie algebra, V a finite–dimensional semisimple $\mathfrak{h}$–module, and $R : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ a meromorphic function. The equation holds in $\mathrm{End}(V \otimes V \otimes V)$, the subscript R_{ij} indicates which tensor factors R acts on, and the symbol $\mathsf{h}^{(j)}$ denotes the weight of the j^{th} tensor factor.

Felder also obtained an elliptic solution $R_F(u, \lambda)$ of the above equations when $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices in $\mathfrak{g} = \mathfrak{gl}_n$, and $V = \mathbb{C}^n$ is the defining representation of $\mathfrak{g}$. He also pointed out that the Knizhnik–Zamolodchikov–Barnard equations satisfied by genus 1 conformal blocks [25] give rise to solutions of the underlying classical dynamical Yang–Baxter equation for any semisimple $\mathfrak{g}$, thus bypassing in a sense Belavin and Drinfeld's no go theorem.

Starting from Felder's discovery, a large number of results have been obtained for elliptic quantum groups. These can be roughly grouped into four related threads described in 1.12–1.15.

1.12. **Felder's elliptic representation category.** The RTT formalism of Faddeev–Reshetikhin–Takhtajan [21], applied to Felder's R–matrix yields a tensor category of representations of $R_F(u, \lambda)$, which can be thought of as the category of representations of a face–type elliptic algebra associated to $\mathfrak{g} = \mathfrak{sl}_n$. This study of this category was initiated in [23, 24] for $\mathfrak{g} = \mathfrak{sl}_2$. An extension of this approach to other Lie types seems, however, less convenient for other classical and for exceptional Lie algebras.

1.13. **Quasi–Hopf twists.** Following the suggestion of Babelon–Bernard–Billey to use dynamical twists to produce solutions of the DQYBE [2], Frønsdal discovered that both the Baxter–Belavin and Felder R–matrices can be obtained from the trigonometric R–matrix of the quantum affine algebra $\widehat{\mathfrak{sl}_n}$ by using such a twist [26, 27]. Frønsdal's analysis was completed by Jimbo–Konno–Odake–Shiraishi [33] and Etingof–Schiffmann [16, 17], who showed that the R–matrix of the quantum affine algebra $\widehat{\mathfrak{g}}$ of any simple Lie algebra $\mathfrak{g}$ can be canonically twisted to an elliptic solution of the QDYBE, thus producing the first examples of such solutions outside of type A. In particular, this suggested defining the elliptic quantum group $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ corresponding to $\mathfrak{g}$ as the quantum affine algebra of $\mathfrak{g}$, with a twisted coproduct and R–matrix. Theorems 1.6 and 1.8 suggest that such an extrinsic definition of $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ may only be appropriate when the parameter parameters $\hbar, \tau$ are formal since, for numerical $\hbar, \tau$, the category of finite–dimensional representations of $U_q(L\mathfrak{g})$ is a covering of that of those of $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$.

1.14. **The elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$.** A related elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ was introduced by Konno [37] for $\mathfrak{g} = \mathfrak{sl}_2$, and any simple $\mathfrak{g}$ by Jimbo–Konno–Odake–Shiraishi [33] as an extension of the quantum affine algebra $U_q\widehat{\mathfrak{g}}$ by a one–dimensional Heisenberg algebra. A presentation of $U_{q,p}(\widehat{\mathfrak{g}})$ in terms of elliptic

Drinfeld (full) currents was obtained by Kojima–Konno for $\mathfrak{g} = \mathfrak{sl}_n$ [35], and by Farghly–Konno–Oshima for an arbitrary $\mathfrak{g}$ [22].

The presentation in [22] is closely related to that given in terms of half–currents in Section 2. Indeed, if one regards the odd theta function $\theta(u)$ as a formal series in the variable $p = e^{2\pi\iota\tau}$ whose coefficients are Laurent polynomials in $z = e^{2\pi\iota u}$, the commutation relations given in Section 2.3 can be formally expanded to get those given in [22, Defn. 2.1]. The converse implication does not appear to be so straightforward, however.

lt is worth pointing out that the definition and presentation of $U_{q,p}(\widehat{\mathfrak{g}})$ are only valid when q and p are formal, whereas the classification of finite–dimensional irreducible representations does not hold in the formal setting since $\mathbb{C}[[\hbar, \tau]]$ is not algebraically closed. This is the case even for the quantum loop algebra $U_q(L\mathfrak{g})$ when $q = e^{\pi\iota\hbar}$, with $\hbar$ formal, where the classification via Drinfeld polynomials fails to hold.

1.15. **Dynamical quantum groups.** It is natural to ask about the algebraic structure of the dynamical quantum groups defined by the DQYBE (1.5). In the absence of a spectral parameter u, these are *Hopf algebroids*, and their study was undertaken up by Etingof–Varchenko [18, 19, 20] (see also [15] for an exposition). A similar framework that would also encompass the spectral parameter u is lacking at present.

1.16. **Geometric representation theory.** There has been a resurgence of interest in elliptic quantum groups recently, especially in connection to the geometry of Nakajima quiver varieties. Recall that, for $\mathfrak{g}$ simply–laced, the Yangian (resp. quantum loop algebra) of $\mathfrak{g}$ acts on the (equivariant) cohomology (resp. K–theory) of the Nakajima quiver varieties corresponding to the Dynkin diagram of $\mathfrak{g}$ [40, 44]. Maulik–Okounkov [38] obtained a new construction of Yangians via stable envelopes in the equivariant cohomology of these quiver varieties (not necessarily of ADE type). This construction has been extended to the elliptic setting by Aganagic–Okounkov [1], and leads to a geometric definition of their category of representations. We also note that a sheafified version of elliptic quantum groups can be obtained by the cohomological Hall algebra construction of Yang–Zhao [46], which sheds light on the type of algebraic objects they are.

1.17. **Outline of the paper.** In Section 2, we define the category $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$, and its full subcategory $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$, for any symmetrisable Kac–Moody algebra $\mathfrak{g}$. Section 3 is devoted to proving a few basic properties of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. Its main result is an analogue of the triangular decomposition (Theorem 3.8). In Section 4, we prove that the highest–weight of an irreducible object of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is of the form (1.2), and also establish a version of Knight's lemma. In Section 5, we review the definition of the quantum loop algebra $U_q(L\mathfrak{g})$, and the classification of the simple objects in $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ in terms of Drinfeld polynomials. Section 6 gives the construction of the functor $\Theta : \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g})) \to \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. In Section 7, we give a solution to the factorization problem for (abelian) doubly quasi–periodic functions. We use this factorization in Section 8 to construct a right inverse to the functor Θ. The final Section 9 gives the classification of irreducible objects in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$.

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2. Elliptic quantum groups

2.1. Let $\mathbf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ be a symmetrisable generalized Cartan matrix [34]. Thus, $a_{ii} = 2$ for any $i \in \mathbf{I}$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ for any $i \neq j \in \mathbf{I}$, and there exists a diagonal matrix D with positive integer entries $\{d_i\}_{i\in\mathbf{I}}$ such that $D\mathbf{A}$ is symmetric. We assume that (d_i) are relatively prime.

Let $(\mathfrak{h}, \{\alpha_i\}_{i\in\mathbf{I}}, \{\alpha_i^\vee\}_{i\in\mathbf{I}})$ be the unique realization of $\mathbf{A}$. Thus, $\mathfrak{h}$ is a complex vector space of dimension $2|\mathbf{I}|-\mathrm{rank}(\mathbf{A})$, $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee\}_{i\in\mathbf{I}} \subset \mathfrak{h}$ are linearly independent sets and, for any $i, j \in \mathbf{I}$, $\alpha_j(\alpha_i^\vee) = a_{ij}$. Let $(.,.) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ denote the non–degenerate symmetric bilinear form satisfying

$$(\lambda, \alpha_i) = d_i\lambda(\alpha_i^\vee), \ \forall\lambda \in \mathfrak{h}^*, i \in \mathbf{I}$$

Let $\mathfrak{g}$ be the Kac–Moody algebra associated to $\mathbf{A}$.

2.2. From now onwards, we fix a complex number $\tau \in \mathbb{H}$ in the upper half plane. Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$ and let $p = \exp(2\pi\iota\tau)$. Let $\theta(u)$ be the odd theta function: namely the unique holomorphic function satisfying the following conditions:

- $\theta(u + 1) = -\theta(u)$
- $\theta(u + \tau) = -e^{-\pi\iota\tau}e^{-2\pi\iota u}\theta(u)$
- $\theta(u) = 0 \iff u \in \Lambda_\tau$
- $\theta'(0) = 1$

For this theta function, we have $\theta(-u) = -\theta(u)$. Explicitly, we have the following expression of $\theta(u)$

$$\theta(u) = -\frac{e^{-\pi\iota u}}{2\pi\iota}(z; p)_\infty(pz^{-1}; p)_\infty(p; p)_\infty^{-2} \tag{2.1}$$

where $z = e^{2\pi\iota u}$ and we use the standard q–Pochhamer notation $(z; p)_\infty = \prod_{k\geq 0}(1-zp^k)$. For future purposes, we will use an additive notation. Define

$$\theta^\pm(u) := \prod_{n\geq 1}\left(1 - p^n e^{\pm 2\pi\iota u}\right) \tag{2.2}$$

Note that $\theta^+(u) = \theta^-(-u) = (pz; p)_\infty$ where $z = e^{2\pi\iota u}$. Moreover we have the following identity:

$$\theta(u) = \sin(\pi u)\frac{\theta^+(u)\theta^-(u)}{\theta^+(0)^2} \tag{2.3}$$

In subsequent calculations the following identity will be extensively used (called Fay's trisecant identity)

$$\theta(a-c)\theta(a+c)\theta(b-d)\theta(b+d) = \theta(a-b)\theta(a+b)\theta(c-d)\theta(c+d) + \theta(a-d)\theta(a+d)\theta(b-c)\theta(b+c) \tag{FTI}$$

2.3. **The category** $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$**.** We now define an elliptic representation category $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$. Let $\hbar \in \mathbb{C}$ be generic in the sense that $\mathbb{Z}\hbar \cap \Lambda_\tau = \{0\}$.

Objects. An object of $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$ is an $\mathfrak{h}$–diagonalizable module $\mathbb{V}$ with finite–dimensional weight space $\mathbb{V} = \oplus_{\mu\in\mathfrak{h}^*}\mathbb{V}_\mu$ together with meromorphic $\mathrm{End}(\mathbb{V})$–valued functions $\{\Phi_i(u), \mathfrak{X}_i^\pm(u,\lambda)\}_{i\in\mathbf{I}}$ of a spectral variable $u \in \mathbb{C}$ and a dynamical variable $\lambda \in \mathfrak{h}^*$, satisfying the following list of axioms:

Category $\mathcal{O}$ and integrability condition.

- There exist $\mu_1, \cdots, \mu_r \in \mathfrak{h}^*$ such that $\mathbb{V}_\mu \neq 0$ implies that $\mu < \mu_k$ for some $k = 1, \cdots, r$.
- For each $\mu \in \mathfrak{h}^*$ such that $\mathbb{V}_\mu \neq 0$ and $i \in \mathbf{I}$, there exists $N > 0$ such that $\mathbb{V}_{\mu-n\alpha_i} = 0$ for all $n \geq N$.

Periodicity conditions.

- $\Phi_i(u+1) = \Phi_i(u)$ and $\Phi_i(u+\tau) = e^{-2\pi\iota\hbar d_i\alpha_i^\vee}\Phi_i(u)$.
- $\mathfrak{X}_i^\pm(u+1,\lambda) = \mathfrak{X}_i(u,\lambda)$ and $\mathfrak{X}_i^\pm(u+\tau,\lambda) = e^{-2\pi\iota(\lambda,\alpha_i)}\mathfrak{X}_i^\pm(u,\lambda)$
- $\mathfrak{X}_i^\pm(u,\lambda+\gamma) = \mathfrak{X}_i^\pm(u,\lambda)$ for every $\gamma \in P^\vee$ where
$$P^\vee := \{\gamma \in \mathfrak{h}^* | (\gamma,\alpha_i) \in \mathbb{Z} \text{ for every } i \in \mathbf{I}\}$$
- Let $D_i^\pm \subset \mathbb{C}\times\mathfrak{h}^*$ be the set of poles of $\mathfrak{X}_i^\pm(u,\lambda)$. Then we require that $D_i^\pm$ is stable under shifts by $\Lambda_\tau \times (P^\vee + \tau P^\vee)$.

Note that as a consequence of the $P^\vee$–periodicity imposed above, the function $\mathfrak{X}_i^\pm(u,\lambda)$ only depends on the variables u and $\lambda_j = (\lambda,\alpha_j)$ where $j \in \mathbf{I}$.

Additional axiom

- Let us define $Q^0 := \{\beta \in Q : (\beta,\alpha_i) = 0\ \forall i \in \mathbf{I}\}$ and let
$$\overline{\mathfrak{h}^*} = \bigcap_{\beta\in Q^0} \mathrm{Ker}((\beta,\cdot)) \subset \mathfrak{h}^*$$
We require that $\mathfrak{X}_i^\pm(u,\lambda)$ is a meromorphic function on $\mathbb{C}\times\overline{\mathfrak{h}^*}$. One can easily prove that [34, Prop. 1.6] $\overline{\mathfrak{h}^*} = \sum_{i\in\mathbf{I}}\mathbb{C}\alpha_i$.

Commutation relations.

($\mathcal{EQ}1$) For each $i,j \in \mathbf{I}$ and $h \in \mathfrak{h}$ we have $[h,\Phi_i(u)] = 0$ and
$$\Phi_i\left(u,\lambda+\frac{\hbar}{2}\alpha_j\right)\Phi_j\left(v,\lambda-\frac{\hbar}{2}\alpha_i\right) = \Phi_j\left(v,\lambda+\frac{\hbar}{2}\alpha_i\right)\Phi_i\left(v,\lambda-\frac{\hbar}{2}\alpha_j\right)$$
Moreover, we assume that $\det(\Phi_i(u,\lambda)) \not\equiv 0$ for each $i \in \mathbf{I}$.

($\mathcal{EQ}2$) For each $i \in \mathbf{I}$ and $h \in \mathfrak{h}$ we have
$$[h,\mathfrak{X}_i^\pm(u,\lambda)] = \pm\alpha_i(h)\mathfrak{X}_i^\pm(u,\lambda)$$

($\mathcal{EQ}3$) For each $i,j\in\mathbf{I}$ let $a=d_ia_{ij}\hbar/2$ and let $\lambda_j=(\lambda,\alpha_j)$. Then the following relation holds on $\mathbb{V}_\mu$.

$$\Phi_i\left(u,\pm\left(\lambda-\frac{\hbar}{2}\mu\right)+\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_j^\pm(v,\lambda)\Phi_i\left(u,\pm\left(\lambda-\frac{\hbar}{2}\mu\right)+\frac{\hbar}{2}(\alpha_i-\alpha_j)\right)^{-1}$$
$$=\frac{\theta(u-v\pm a)}{\theta(u-v\mp a)}\mathfrak{X}_j^\pm(v,\lambda\pm\hbar\alpha_i)\pm\frac{\theta(2a)\theta(u-v\mp a-\lambda_j)}{\theta(\lambda_j)\theta(u-v\mp a)}\mathfrak{X}_j^\pm(u\mp a,\lambda\pm\hbar\alpha_i)$$

($\mathcal{EQ}4$) For each $i,j\in\mathbf{I}$ and $\lambda\in\mathfrak{h}^*$, let $a=d_ia_{ij}\hbar/2$. Then we have

$$\begin{aligned}
&\theta(\lambda_i+\lambda_j)\theta(u-v\mp a)\mathfrak{X}_i^\pm\left(u,\lambda\pm\frac{\hbar}{2}\alpha_j\right)\mathfrak{X}_j^\pm\left(v,\lambda\mp\frac{\hbar}{2}\alpha_i\right)\\
&\quad-\theta(\lambda_i\pm a)\theta(u-v-\lambda_j)\mathfrak{X}_i^\pm\left(u,\lambda\pm\frac{\hbar}{2}\alpha_j\right)\mathfrak{X}_j^\pm\left(u+\lambda_i,\lambda\mp\frac{\hbar}{2}\alpha_i\right)\\
&\quad-\theta(\lambda_j\mp a)\theta(u-v+\lambda_i)\mathfrak{X}_i^\pm\left(v+\lambda_j,\lambda\pm\frac{\hbar}{2}\alpha_j\right)\mathfrak{X}_j^\pm\left(v,\lambda\mp\frac{\hbar}{2}\alpha_i\right)\\
&\quad=\theta(\lambda_i+\lambda_j)\theta(u-v\pm a)\mathfrak{X}_j^\pm\left(v,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^\pm\left(u,\lambda\mp\frac{\hbar}{2}\alpha_j\right)\\
&\quad-\theta(\lambda_i\mp a)\theta(u-v-\lambda_j)\mathfrak{X}_j^\pm\left(u+\lambda_i,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^\pm\left(u,\lambda\mp\frac{\hbar}{2}\alpha_j\right)\\
&\quad-\theta(\lambda_j\pm a)\theta(u-v+\lambda_i)\mathfrak{X}_j^\pm\left(v,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^\pm\left(v+\lambda_j,\lambda\mp\frac{\hbar}{2}\alpha_j\right)
\end{aligned}$$

where $\lambda_i=(\lambda,\alpha_i)$ and $\lambda_j=(\lambda,\alpha_j)$.

($\mathcal{EQ}5$) For each $i,j\in\mathbf{I}$ and $\lambda_1,\lambda_2\in\mathfrak{h}^*$ such that $\lambda_1+\lambda_2=\hbar(\mu+\alpha_i-\alpha_j)$, the following relation holds on the weight space $\mathbb{V}_\mu$, with $\lambda=\lambda_1-\hbar(\mu-\alpha_i+\alpha_j)/2$.

$$\theta(d_i\hbar)[\mathfrak{X}_i^+(u,\lambda_1),\mathfrak{X}_j^-(v,\lambda_2)]=\delta_{ij}\left(\frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\Phi_i(v,\lambda)+\frac{\theta(u-v-\lambda_{2,i})}{\theta(u-v)\theta(\lambda_{2,i})}\Phi_i(u,\lambda)\right)$$

where $\lambda_{s,i}=(\lambda_s,\alpha_i)$ for $s=1,2$.

Morphisms. A morphism between two objects $\mathbb{V}$ and $\mathbb{W}$ of $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$ is a meromorphic function $\varphi:\mathfrak{h}^*\to\mathrm{Hom}_{\mathfrak{h}}(\mathbb{V},\mathbb{W})$, such that

- For each $i\in\mathbf{I}$ we have
$$\varphi\left(\lambda+\frac{\hbar}{2}\alpha_i\right)\Phi_i(u,\lambda)=\Phi_i(u,\lambda)\varphi\left(\lambda-\frac{\hbar}{2}\alpha_i\right)$$
- For each $i\in\mathbf{I}$ the following equation holds in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{V}_\mu,\mathbb{W}_{\mu\pm\alpha_i})$

$$\varphi\left(\pm\lambda\mp\frac{\hbar}{2}(\mu\pm\alpha_i)+\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^\pm(u,\lambda)=\mathfrak{X}_i^\pm(u,\lambda)\varphi\left(\pm\lambda\mp\frac{\hbar}{2}\mu-\frac{\hbar}{2}\alpha_i\right)$$

The composition of morphisms is defined as $(\varphi\circ\psi)(\lambda)=\varphi(\lambda)\psi(\lambda)$. The constant function $\mathbf{1}_{\mathbb{V}}(\lambda)=\mathrm{Id}_{\mathbb{V}}$ is the identity morphism in $\mathrm{Hom}_{\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))}(\mathbb{V},\mathbb{V})$.

Remark. The definition of a morphism given above differs from the one usually given for RLL type algebras, for instance in [15, 20, 24]. The reason is the freedom in choosing this notion: given a set–theoretic map $\beta:\mathfrak{h}^*\to\mathfrak{h}^*$ and a zero–weight meromorphic function $\varphi(\lambda) \in \mathrm{Aut}(\mathbb{V})$ the reader can easily verify that the following shifted conjugation respects the relations ($\mathcal{EQ}1$)–($\mathcal{EQ}5$):

$$\begin{aligned}\widetilde{\Phi_i}(u,\lambda) &= \varphi\left(\lambda + \frac{\hbar}{2}\alpha_i\right)\Phi_i(u,\lambda)\varphi\left(\lambda - \frac{\hbar}{2}\alpha_i\right)^{-1}\\ \widetilde{\mathfrak{X}_i}^{\pm}(u,\lambda) &= \varphi\left(\pm\lambda + \beta(\mu \pm \alpha_i) \mp \frac{\hbar}{2}(\mu \pm \alpha_i) + \frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}(u,\lambda)\\ &\qquad .\varphi\left(\pm\lambda + \beta(\mu) \mp \frac{\hbar}{2}\mu - \frac{\hbar}{2}\alpha_i\right)^{-1}\end{aligned}$$

Clearly any two choices are related by an affine linear change of dynamical variables. The one given above corresponds to $\beta \equiv 0$. From the RLL picture one arrives at $\beta(\mu) = \pm\frac{\hbar}{2}\mu$, depending on the choice of Gauss decomposition of the L–matrix.

2.4. For $\mathfrak{g} = \mathfrak{sl}_2$ the category defined above is same as the one defined by Felder [23] and further studied by Felder and Varchenko [24]. The reader can consult [14] for a proof of this assertion. More generally, the previous statement holds for $\mathfrak{g} = \mathfrak{sl}_n$. A proof of this will be given in [28]. Recently, similar calculation appeared in [36] where the formal case is treated (*i.e.*, $p = e^{2\pi\iota\tau}$ is considered a formal variable).

We also note that we have not imposed any Serre-type relations in the definition above. This is in accordance with the assertion that for Yangians and quantum loop algebras, the Serre relations are a consequence of the other relations and category $\mathcal{O}$ and integrability axiom [30, Propositions 2.7, 2.13].

2.5. As mentioned in the introduction, we restrict our attention to the full subcategory $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ of $\mathrm{Rep}(\mathcal{E}_{\hbar,\tau}(\mathfrak{g}))$ consisting of objects $(\mathbb{V}, \Phi_i(u,\lambda), \mathfrak{X}_i(u,\lambda))$ such that $\Phi_i(u,\lambda)$ is independent of λ, for each $i \in \mathbf{I}$. The relations ($\mathcal{EQ}1$) and ($\mathcal{EQ}3$) of the previous section become simpler for such objects as follows:

($\mathcal{EQ}1$) For each $i,j \in \mathbf{I}$ and $h \in \mathfrak{h}$ we have

$$[\Phi_i(u), \Phi_j(v)] = 0 \qquad \text{and} \qquad [h, \Phi_i(u)] = 0$$

($\mathcal{EQ}3$) For each $i,j \in \mathbf{I}$ let $a = d_i a_{ij}\hbar/2$ and let $\lambda_j = (\lambda, \alpha_j)$. Then

$$\begin{aligned}\Phi_i(u)\mathfrak{X}_j^{\pm}(v,\lambda)\Phi_i(u)^{-1} &= \frac{\theta(u-v\pm a)}{\theta(u-v\mp a)}\mathfrak{X}_j^{\pm}(v,\lambda\pm\hbar\alpha_i)\\ &\qquad \pm\frac{\theta(2a)\theta(u-v\mp a-\lambda_j)}{\theta(\lambda_j)\theta(u-v\mp a)}\mathfrak{X}_j^{\pm}(u\mp a,\lambda\pm\hbar\alpha_i)\end{aligned}$$

2.6. From the relation ($\mathcal{EQ}3$) we obtain, by setting $v = u \pm a$

$$\mathrm{Ad}(\Phi_i(u))\mathfrak{X}_j^{\pm}(u\pm a,\lambda) = \frac{\theta(\lambda_j \pm 2a)}{\theta(\lambda_j)}\mathfrak{X}_j^{\pm}(u\mp a,\lambda\pm\hbar\alpha_i)$$

Using this we obtain the following from ($\mathcal{EQ}3$) by taking $\mathrm{Ad}(\Phi_i(u))^{-1}$ on the both sides, and replacing λ by $\lambda \mp \hbar\alpha_i$:

($\mathcal{EQ}3'$) For every $i,j\in\mathbf{I}$ the following holds

$$\Phi_i(u)^{-1}\mathfrak{X}_j^{\pm}(v,\lambda)\Phi_i(u)=\frac{\theta(u-v\mp a)}{\theta(u-v\pm a)}\mathfrak{X}_j^{\pm}(v,\lambda\mp\hbar\alpha_i)$$
$$\mp\frac{\theta(2a)\theta(u-v\pm a-\lambda_j)}{\theta(\lambda_j)\theta(u-v\pm a)}\mathfrak{X}_j^{\pm}(u\pm a,\lambda\mp\hbar\alpha_i)$$

Similar to the case of Yangians and quantum loop algebras (see [30, Remarks 2.3, 2.10]), the relation ($\mathcal{EQ}4$) can be simplified when $i=j$ as follows. Let $u=v$ in ($\mathcal{EQ}4$) for $i=j$ and hence $a=d_i\hbar$. We get

$$\theta(\lambda_i\pm d_i\hbar)\mathfrak{X}_i^{\pm}\left(u,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(u+\lambda_i,\lambda\mp\frac{\hbar}{2}\alpha_i\right)$$
$$-\theta(\lambda_i\mp d_i\hbar)\mathfrak{X}_i^{\pm}\left(u+\lambda_i,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(u,\lambda\mp\frac{\hbar}{2}\alpha_i\right)$$
$$=\pm\frac{\theta(d_i\hbar)\theta(2\lambda_i)}{\theta(\lambda_i)}\mathfrak{X}_i^{\pm}\left(u,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(u,\lambda\mp\frac{\hbar}{2}\alpha_i\right)\quad(2.4)$$

This implies the following relation again from ($\mathcal{EQ}4$)

($\mathcal{EQ}4'$) For each $i\in\mathbf{I}$ and $\lambda\in\mathfrak{h}^*$ we have

$$\theta(u-v\mp d_i\hbar)\mathfrak{X}_i^{\pm}\left(u,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(v,\lambda\mp\frac{\hbar}{2}\alpha_i\right)$$
$$\pm\frac{\theta(u-v-\lambda_i)\theta(d_i\hbar)}{\theta(\lambda_i)}\mathfrak{X}_i^{\pm}\left(u,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(u,\lambda\mp\frac{\hbar}{2}\alpha_i\right)=$$
$$\theta(u-v\pm d_i\hbar)\mathfrak{X}_i^{\pm}\left(v,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(u,\lambda\mp\frac{\hbar}{2}\alpha_i\right)$$
$$\pm\frac{\theta(u-v+\lambda_i)\theta(d_i\hbar)}{\theta(\lambda_i)}\mathfrak{X}_i^{\pm}\left(v,\lambda\pm\frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}\left(v,\lambda\mp\frac{\hbar}{2}\alpha_i\right)$$

In this form the relation ($\mathcal{EQ}4$) appeared for $\mathfrak{g}=\mathfrak{sl}_2$ in [14, Prop. 1.1].

3. Some preparatory results

The aim of this section is to prove a weak triangularity property for irreducible objects of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. In order to do so we will introduce the notion of a λ–flat object and prove that every $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is isomorphic to some λ–flat $\mathbb{V}^\flat$. The triangularity property holds for $\mathbb{V}^\flat$, as stated in Theorem 3.8.

3.1. Poles of $\mathfrak{X}_j^{\pm}(u,\lambda)$.

As a consequence of the periodicity axioms of section 2.3, we have the following

Proposition. *The set of poles of $\mathfrak{X}_j^{\pm}(u,\lambda)$ are contained in the union of affine hyperplanes of the form $u=b$ ($b\in\mathbb{C}$) or $\lambda_j=(\lambda,\alpha_j)\in\Lambda_\tau$. Moreover the poles at $\lambda_j\in\Lambda_\tau$ are of order at most 1.*

Proof. Let us restrict to the $+$ case for definiteness. Let $D_i^+ \subset \mathbb{C} \times \mathfrak{h}^*$ be the set of poles of $\mathfrak{X}_i^+(u,\lambda)$. Consider a point $(v^0, \lambda^0) \in D_i^+$.

Claim: Either $(v^0,\lambda) \in D_i^+$ for every $\lambda \in \mathfrak{h}^*$ or $D_i^+ \cap \{(v^0, \lambda^0 + p\hbar\alpha_i)\}_{p\in\mathbb{Z}}$ is finite.

Proof of the claim: Assume that the set $D_i^+ \cap \{(v^0, \lambda^0 + p\hbar\alpha_i)\}_{p\in\mathbb{Z}}$ is infinite. Since D_i^+ is stable under shifts by $P^\vee + \tau P^\vee$ in the dynamical variable, and $p\hbar\alpha_i \in P^\vee + \tau P^\vee$ implies $p = 0$ by the genericity assumption on $\hbar$, the set $\lambda^0 + \mathbb{Z}\hbar\alpha_i + (P^\vee + \tau P^\vee)$ is dense in $\mathfrak{h}^*$. Hence (v^0,λ) is in D_i^+ for every $\lambda \in \mathfrak{h}^*$.

In order to prove the proposition we will restrict our attention to the later case. Namely, there exist $n, m \in \mathbb{N}$ such that

$$D_i^+ \cap \{(v^0, \lambda^0 + p\hbar\alpha_i)\}_{p\in\mathbb{Z}} \subset \{(v^0, \lambda^0 - m\hbar\alpha_i), \ldots, (v^0, \lambda^0 + n\hbar\alpha_i)\}$$

Let $\lambda^s = \lambda^0 - m\hbar\alpha_i$ and $\lambda^t = \lambda^0 + n\hbar\alpha_i$. Without loss of generality we may assume $(v^0,\lambda^s), (v^0,\lambda^t) \in D_i^+$. Consider the relation $(\mathcal{EQ}3)$ with $j = i$ and $v = v^0$:

$$\begin{aligned}\mathrm{Ad}(\Phi_i(u))\mathfrak{X}_i^+(v^0,\lambda) = {} & \frac{\theta(u - v^0 + d_i\hbar)}{\theta(u - v^0 - d_i\hbar)}\mathfrak{X}_i^+(v^0, \lambda + \hbar\alpha_i) \\ & + \frac{\theta(d_i\hbar)\theta(u - v^0 - \lambda_i - d_i\hbar)}{\theta(u - v^0 - d_i\hbar)\theta(\lambda_i)}\mathfrak{X}_i^+(u - d_i\hbar, \lambda + \hbar\alpha_i)\end{aligned}$$

Since the left–hand side has a pole at $\lambda = \lambda^t$, so must the right–hand side . However $(v^0, \lambda^t + \hbar\alpha_i) \notin D_i^+$. Therefore we have $\lambda_i^t \in \Lambda_\tau$. A similar argument using $(\mathcal{EQ}3$') from Section 2.6 implies that $\lambda_i^s \in \Lambda_\tau$. Therefore we get

$$(n+m)\hbar(2d_i) = (\lambda^s - \lambda^t, \alpha_i) \in \Lambda_\tau$$

which implies that $n = m = 0$. Therefore $\lambda_i^0 = M - N\tau \in \Lambda_\tau$.

Again consider the relation $(\mathcal{EQ}3)$

$$\begin{aligned}\mathrm{Ad}(\Phi_i(u))\left(\theta(\lambda_i)\mathfrak{X}_i^+(v,\lambda)\right) = {} & \frac{\theta(u - v + d_i\hbar)}{\theta(u - v - d_i\hbar)}\theta(\lambda_i)\mathfrak{X}_i^+(v, \lambda + \hbar\alpha_i) \\ & + \frac{\theta(d_i\hbar)\theta(u - v - \lambda_i - d_i\hbar)}{\theta(u - v - d_i\hbar)}\mathfrak{X}_i^+(u - d_i\hbar, \lambda + \hbar\alpha_i)\end{aligned}$$

The right–hand side can be specialized at $\lambda = \lambda^0$ and gives

$$\text{R.H.S. } |_{\lambda=\lambda^0} = (-1)^{M+N} e^{-\pi\iota N^2\tau} e^{-2\pi\iota N(u-v-d_i\hbar)}\theta(d_i\hbar)\mathfrak{X}_i^+(u - d_i\hbar, \lambda^0 + \hbar\alpha_i)$$

Thus we get

$$\begin{aligned}\theta(\lambda_i)\mathfrak{X}_i^+(v,\lambda)\big|_{\lambda=\lambda^0} = {} & e^{2\pi\iota Nv}\,\mathrm{Ad}(\Phi_i(u))^{-1}\cdot \\ & \cdot\Big((-1)^{M+N} e^{-\pi\iota N^2\tau} e^{-2\pi\iota N(u-d_i\hbar)}\theta(d_i\hbar)\mathfrak{X}_i^+(u - d_i\hbar, \lambda^0 + \hbar\alpha_i)\Big)\end{aligned}$$

The right–hand side is either identically zero or nowhere zero. The former case will contradict the fact that $(v^0, \lambda^0) \in D_i^+$ since $\theta(x)$ has only simple zeroes. In the later case we get that $(v, \lambda^0) \in D_i^+$ for every v and the corresponding pole is simple.

□

3.2. **Partial fractions.** Let $S_j^\pm \subset \mathbb{C}$ be a choice of representatives modulo Λ_τ of poles of $\mathfrak{X}_j^\pm(u,\lambda)$ in the spectral variable u. Consider the Laurent series expansion of these functions near $b \in S_j^\pm$:

$$\mathfrak{X}_j^\pm(u,\lambda) = \sum_{n\in\mathbb{N}} \mathsf{X}_{j;b,n}^\pm(\lambda)\frac{1}{(u-b)^{n+1}} + \text{ part holomorphic near } b$$

Note that the sum appearing above need not be finite (since $\mathbb{V}$ is infinite–dimensional in general). However, for each weight $\mu \in \mathfrak{h}^*$, the sum is finite in $\mathrm{Hom}_\mathbb{C}(\mathbb{V}_\mu, \mathbb{V}_{\mu\pm\alpha_j})$. We have the following analogue of partial fraction decomposition of a rational function:

Lemma. *The* $\mathrm{End}(\mathbb{V})$*–valued functions* $\mathfrak{X}_j^\pm(u,\lambda)$ *have the following form, where* $\mathsf{X}_{j;b,n}^\pm(\lambda)$ *are holomorphic functions of* λ

$$\mathfrak{X}_j^\pm(u,\lambda) = \sum_{\substack{b\in S_j^\pm\\ n\in\mathbb{N}}} \mathsf{X}_{j;b,n}^\pm(\lambda)\frac{(-\partial_u)^n}{n!}\left(\frac{\theta(u-b+\lambda_j)}{\theta(u-b)\theta(\lambda_j)}\right)$$

Proof. The proof of this lemma is fairly standard. We take the difference of the two sides of the equation given above

$$F(u,\lambda) = \mathfrak{X}_j^\pm(u,\lambda) - \sum_{\substack{b\in S_j^\pm\\ n\in\mathbb{N}}} \mathsf{X}_{j;b,n}^\pm(\lambda)\frac{(-\partial_u)^n}{n!}\left(\frac{\theta(u-b+\lambda_j)}{\theta(u-b)\theta(\lambda_j)}\right)$$

Then this function is holomorphic in u and has the quasi–periodicity: $F(u+1,\lambda) = F(u,\lambda)$ and $F(u+\tau,\lambda) = e^{-2\pi\iota\lambda_j}F(u,\lambda)$. Since there are no such holomorphic functions, other than 0, we get the desired equation. Finally the claim that $\mathsf{X}_{j;b,n}^\pm(\lambda)$ are holomorphic functions of λ follows from Proposition 3.1 which states that $\theta(\lambda_j)\mathfrak{X}_j^\pm(u,\lambda)$ has no poles in λ variable. □

3.3. **A difference equation.** As a consequence of the commutation relation ($\mathcal{EQ}3$) we get the following equation for the functions $\mathsf{X}_{j;b,n}^\pm(\lambda)$ appearing in Lemma 3.2 above.

Lemma. *For each* $i,j \in \mathbf{I}$*,* $n \in \mathbb{N}$ *and* $b \in S_j^\pm$ *we have*

$$\mathrm{Ad}(\Phi_i(u)).\mathsf{X}_{j;b,n}^\pm(\lambda) = \sum_{k\geq 0}\frac{\partial_v^k}{k!}\left(\frac{\theta(u-v\pm a)}{\theta(u-v\mp a)}\right)\bigg|_{v=b}\mathsf{X}_{j;b,n+k}^\pm(\lambda\pm\hbar\alpha_i) \tag{3.1}$$

where again $a = \hbar d_i a_{ij}/2$.

Proof. Consider the commutation relation ($\mathcal{EQ}3$):

$$\begin{aligned}\Phi_i(u)\mathfrak{X}_j^\pm(v,\lambda)\Phi_i(u)^{-1} = \frac{\theta(u-v\pm a)}{\theta(u-v\mp a)}&\mathfrak{X}_j^\pm(v,\lambda\pm\hbar\alpha_i)\\ &\pm\frac{\theta(2a)\theta(u-v\mp a-\lambda_j)}{\theta(\lambda_j)\theta(u-v\mp a)}\mathfrak{X}_j^\pm(u\mp a,\lambda\pm\hbar\alpha_i)\end{aligned}$$

If we multiply both sides of this equation by $(v-b)^n$ and integrate over a small circle around b, we get the equation (3.1). □

3.4. **Generalized eigenspaces.** Given $\mu \in \mathfrak{h}^*$ define $\mathcal{M}(\mu)$ to be the set of $\mathbf{I}$–tuples of meromorphic functions $\underline{\mathcal{A}} = (A_i(u))$ such that for each $i \in \mathbf{I}$ we have

$$A_i(u+1) = A_i(u) \qquad \text{and} \qquad A_i(u+\tau) = e^{-2\pi\iota\hbar(\mu,\alpha_i)} A_i(u)$$

According to the commutation relation ($\mathcal{EQ}1$), we can decompose each weight space $\mathbb{V}_\mu$ into the generalized eigenspaces for the functions $\{\Phi_i(u)\}$. Define:

$$\mathbb{V}_\mu[\underline{\mathcal{A}}] := \{v \in \mathbb{V}_\mu : \text{ for every } i \in \mathbf{I}, (\Phi_i(u) - A_i(u))^N v = 0 \text{ for } N \gg 0\}$$

Then we have

$$\mathbb{V}_\mu = \bigoplus_{\underline{\mathcal{A}} \in \mathcal{M}(\mu)} \mathbb{V}_\mu[\underline{\mathcal{A}}]$$

3.5.

Proposition. *For a given $\mu \in \mathfrak{h}^*$ and $\underline{\mathcal{A}} \in \mathcal{M}(\mu)$, $\underline{\mathcal{A}}^\pm \in \mathcal{M}(\mu \pm \alpha_j)$ consider the following composition, for $j \in \mathbf{I}$ and $n \in \mathbb{N}$:*

$$\begin{tikzcd}
\mathbb{V}_\mu[\underline{\mathcal{A}}] \arrow[r, dashed, "\mathsf{X}^\pm_{j;b,n}(\lambda)^{\underline{\mathcal{A}}^\pm}_{\underline{\mathcal{A}}}"] \arrow[d, hook] & \mathbb{V}_{\mu\pm\alpha_j}[\underline{\mathcal{A}}^\pm] \\
\mathbb{V}_\mu \arrow[r, "\mathsf{X}^\pm_{j;b,n}(\lambda)"'] & \mathbb{V}_{\mu\pm\alpha_j} \arrow[u, two heads]
\end{tikzcd}$$

If $\mathsf{X}^\pm_{j;b,n}(\lambda)^{\underline{\mathcal{A}}^\pm}_{\underline{\mathcal{A}}} \neq 0$ then there exist $\alpha^\pm \in Q$ such that for every $i \in \mathbf{I}$ we have

$$A_i^\pm(u) = A_i(u) \frac{\theta(u-b\pm a)}{\theta(u-b\mp a)} e^{2\pi\iota\hbar(\alpha_i,\alpha^\pm)}$$

where $a = d_i a_{ij}\hbar/2$. Moreover in this case, for every $m \in \mathbb{N}$ we have

$$\exp\left(\mp 2\pi\iota(\lambda,\alpha^\pm)\right) \mathsf{X}^\pm_{j;b,m}(\lambda) \text{ is independent of } \lambda$$

Proof. For the purposes of the proof, let us keep $j \in \mathbf{I}$ and $b \in \mathbb{C}$ fixed. Consider $W^\pm = \mathrm{Hom}_\mathbb{C}(\mathbb{V}_\mu[\underline{\mathcal{A}}], \mathbb{V}_{\mu\pm\alpha_j}[\underline{\mathcal{A}}^\pm])$ as a finite–dimensional vector space over $\mathbb{C}$. We are given $Y_i(u) = \mathrm{Ad}(\Phi_i(u)) \in \mathrm{End}(W^\pm)$ which are $\mathrm{End}(W^\pm)$–valued meromorphic function of u, such that $Y_i(u) - \frac{A_i^\pm(u)}{A_i(u)}\mathrm{Id}$ is a nilpotent operator.

According to the equation (3.1), we are looking for a collection of holomorphic $W^\pm$–valued functions, $x_0^\pm(\lambda), \cdots, x_N^\pm(\lambda)$ where $\lambda \in \overline{\mathfrak{h}^*}$ which are $P^\vee$–periodic. These functions are required to satisfy: for each $i \in \mathbf{I}$,

$$Y_i(u).x_n^\pm(\lambda) = \sum_{l\geq 0} E_{i,l}^\pm(u) x_{n+l}^\pm(\lambda \pm \hbar\alpha_i) \tag{3.2}$$

where the functions $E_{i,l}^\pm(u)$ are given by

$$E_{i,l}^\pm(u) = \frac{\partial_v^l}{l!}\left(\frac{\theta(u-v\pm a)}{\theta(u-v\mp a)}\right)\Bigg|_{v=b} \text{ where } a = d_i a_{ij}\hbar/2$$

We prove the assertion of the proposition by descending (finite) induction, as follows. Consider the function $x_N^\pm(\lambda)$ and its Laurent series expansion. The role of Laurent monomials is played by (because of $P^\vee$ periodicity) $\exp(2\pi\iota(\lambda,\alpha))$ for $\alpha\in Q$. However, since $x_N^\pm(\lambda)$ is a function on $\overline{\mathfrak{h}^*}$, $(\lambda,\alpha)=0$ for every $\alpha\in Q^0$. We remind the reader that $Q^0=\{\beta\in Q:(\beta,\alpha_i)=0\,\forall i\in\mathbf{I}\}$.

Therefore the Laurent series of $x_N^\pm(\lambda)$ has the following form:

$$x_N^\pm(\lambda)=\sum_{\alpha\in Q/Q^0}C_N^\pm(\alpha)\exp\left(2\pi\iota(\lambda,\alpha)\right)$$

Substituting this expansion in the equation (3.2), and comparing the coefficients of the Laurent series, we get

$$Y_i(u)C_N^\pm(\alpha)=E_{i,0}^\pm(u)\exp\left(\pm 2\pi\iota\hbar(\alpha_i,\alpha)\right)C_N(\alpha)$$

Thus $C_N^\pm(\alpha)\neq 0$ implies that (since the only eigenvalue of $Y_i(u)$ on $W^\pm$ is $A_i^\pm(u)/A_i(u)$):

$$E_{i,0}^\pm(u)\exp\left(\pm 2\pi\iota\hbar(\alpha_i,\alpha)\right)=\frac{A_i^\pm(u)}{A_i(u)}$$

This proves the first assertion of the proposition. Namely, there must exist $\alpha^\pm\in Q$ such that

$$\frac{A_i^\pm}{A_i}=\frac{\theta(u-b\pm a)}{\theta(u-b\mp a)}e^{2\pi\iota\hbar(\alpha_i,\alpha^\pm)}\text{ where }a=d_ia_{ij}\hbar/2$$

Note that such $\alpha^\pm$ are not unique (unless $\mathbf{A}$ is non–degenerate), but their classes modulo Q^0 are. Thus we will have to make a choice of $\alpha^\pm\in Q$ so that the equation given above holds. Then we get that $x_N^\pm(\lambda)=e^{\pm 2\pi\iota(\lambda,\alpha^\pm)}C_N^\pm$.

Let us assume that we have proved

$$\exp\left(\mp 2\pi\iota(\lambda,\alpha^\pm)\right)x_m^\pm(\lambda)=C_m^\pm\text{ is independent of }\lambda$$

for each m with $n<m\le N$. Now we will prove it for $x_n^\pm(\lambda)$. Again, let us write the Laurent series expansion of $x_n^\pm(\lambda)$ as

$$x_n^\pm(\lambda)=\sum_{\alpha\in Q/Q^0}C_n^\pm(\alpha)\exp\left(2\pi\iota(\lambda,\alpha)\right)$$

We substitute this in the equation (3.2) and compare the coefficients. For each $i\in\mathbf{I}$, we get

$$Y_i(u)C_n^\pm(\alpha)=E_{i,0}^\pm(u)\exp\left(\pm 2\pi\iota\hbar(\alpha_i,\alpha)\right)C_n^\pm(\alpha)+\\ \delta_{\alpha,\alpha^\pm\bmod Q^0}e^{\pm 2\pi\iota\hbar(\alpha_i,\alpha^\pm)}\sum_{l\ge 1}E_{i,l}^\pm(u)C_{n+l}^\pm$$

The same argument as above applies and we are done. □

3.6. **λ–flat representations and first gauge transformation.** Given a weight $\mu \in \mathfrak{h}^*$ of $\mathbb{V}$ consider the set of generalized eigenvalues

$$\mathcal{M}(\mu;\mathbb{V}) := \{\underline{\mathcal{A}} \in \mathcal{M}(\mu) : \mathbb{V}_\mu[\underline{\mathcal{A}}] \neq 0\}$$

In view of Proposition 3.5 above, we introduce an equivalence on the set $\mathcal{M}(\mu;\mathbb{V})$: $\underline{\mathcal{A}} \sim \underline{\mathcal{A}}'$ if, and only if there exists $\alpha \in Q$ such that for every $i \in \mathbf{I}$ we have $A_i'(u) = A_i(u)\exp(2\pi\iota\hbar(\alpha_i,\alpha))$.

Definition. We say $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is λ–flat if for every $\underline{\mathcal{A}}, \underline{\mathcal{A}}' \in \mathcal{M}(\mu;\mathbb{V})$ such that $\underline{\mathcal{A}} \sim \underline{\mathcal{A}}'$ we have $\underline{\mathcal{A}} = \underline{\mathcal{A}}'$.

Proposition. *Every $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is isomorphic to a λ–flat $\mathbb{V}^\flat \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$.*

Proof. We claim that there exists $\psi(\lambda) \in \mathrm{Aut}(\mathbb{V})$ such that upon conjugating with ψ the we obtain a λ–flat object of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. This automorphism ψ is weight preserving: $\psi(\lambda) = \oplus_\mu \psi_\mu(\lambda)$ where $\psi_\mu(\lambda) \in \mathrm{Aut}(\mathbb{V}_\mu)$. Below we give the construction of ψ_μ.

Choose representatives $\{\underline{\mathcal{A}}^{(b)}\}_{b\in B} \subset \mathcal{M}(\mu;\mathbb{V})$ of their respective equivalence classes modulo $\sim$. Let $\mathbb{V}_\mu[\underline{\mathcal{B}}]$ be a generalized eigenspace, where $\underline{\mathcal{B}} \sim \underline{\mathcal{A}}^{(b)}$ for some $b \in B$. Then by definition we have $\alpha \in Q$ such that

$$B_i(u) = A_i^{(b)}(u)e^{2\pi\iota\hbar(\alpha,\alpha_i)}$$

The action of $\psi_\mu(\lambda)$ on the generalized weight space $\mathbb{V}_\mu[\underline{\mathcal{B}}]$ is then given by

$$\psi_\mu(\lambda)|_{\mathbb{V}_\mu[\underline{\mathcal{B}}]} = \exp\left(-2\pi\iota(\lambda,\alpha)\right)\mathrm{Id}_{\mathbb{V}_\mu[\underline{\mathcal{B}}]}$$

Having defined $\psi(\lambda)$, we consider the object $\mathbb{V}^\flat$ of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ obtained by twisting $\mathbb{V}$ via ψ in accordance with the notion of morphisms introduced in Section 2.3. Namely, $\mathbb{V}^\flat = \mathbb{V}$ as an $\mathfrak{h}$–diagonalizable module and the operators $\{\Phi_i^\flat, \mathfrak{X}_i^{\pm,\flat}\}$ on a weight space $\mathbb{V}_\mu^\flat = \mathbb{V}_\mu$ are given by:

$$\begin{aligned}
\Phi_i^\flat(u) &= \varphi\left(\lambda + \frac{\hbar}{2}\alpha_i\right)\Phi_i(u)\varphi\left(\lambda - \frac{\hbar}{2}\alpha_i\right)^{-1} \\
\mathfrak{X}_i^{\pm,\flat}(u,\lambda) &= \varphi\left(\pm\lambda \mp \frac{\hbar}{2}(\mu \pm \alpha_i) + \frac{\hbar}{2}\alpha_i\right)\mathfrak{X}_i^{\pm}(u,\lambda) \\
&\qquad .\varphi\left(\pm\lambda \mp \frac{\hbar}{2}\mu - \frac{\hbar}{2}\alpha_i\right)^{-1}
\end{aligned}$$

The reader can easily verify that, by the construction of ψ, $\mathbb{V}^\flat$ is a λ–flat object. □

3.7. Assume $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is a λ–flat object. Then the following is a consequence of Proposition 3.5.

Lemma. *Let $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ be a λ–flat object. Then for every $\mu \in \mathfrak{h}^*$ a weight of $\mathbb{V}$ the following two subspaces of $\mathbb{V}_{\mu\pm\alpha_j}$ are identical:*

$$\sum_{\substack{u\in\mathbb{C}\\ \lambda\in\mathfrak{h}^*}} \mathfrak{X}_j^{\pm}(u,\lambda)\mathbb{V}_\mu = \sum_{u\in\mathbb{C}} \mathfrak{X}_j^{\pm}(u,\lambda^0)\mathbb{V}_\mu$$

where $\lambda^0 \in \mathfrak{h}^*$ is chosen arbitrarily such that $\lambda_j^0 \notin \Lambda_\tau$.

Proof. According to the partial fraction decompotion of $\mathfrak{X}_j^{\pm}(u,\lambda)$ given in Lemma 3.2, we have

$$\text{Span of } \{\mathfrak{X}_j^{\pm}(u,\lambda)\}_{u\in\mathbb{C}} = \text{ Span of } \{\mathsf{X}_{j;b,n}^{\pm}(\lambda)\}_{b\in S_j^{\pm},n\in\mathbb{N}}$$

where as in Lemma 3.2, $S_j^{\pm}\subset\mathbb{C}$ are a choice of representatives of poles of $\mathfrak{X}_j^{\pm}(u,\lambda)$ in the spectral variable u, modulo Λ_τ.

Let $\underline{\mathcal{A}}\in\mathcal{M}(\mu;\mathbb{V})$ and $b\in S_j^{\pm}$. Define $\underline{\mathcal{A}}^{\pm}$ by:

$$A_i^{\pm}(u) = A_i(u)\frac{\theta(u-b\pm a)}{\theta(u-b\mp a)}$$

Proposition 3.5 implies that

$$\mathsf{X}_{j;b,n}^{\pm}(\lambda)\left(\mathbb{V}_\mu[\underline{\mathcal{A}}]\right)\subset\bigoplus_{\underline{\mathcal{B}}\sim\underline{\mathcal{A}}^{\pm}}\mathbb{V}_{\mu\pm\alpha_j}[\underline{\mathcal{B}}]$$

The right–hand side has only one summand, by the definition of λ–flat object. Thus Proposition 3.5 also gives that $\mathsf{X}_{j;b,n}^{\pm}(\lambda)$ on $\mathbb{V}_\mu[\underline{\mathcal{A}}]$ has the form $e^{2\pi\iota(\lambda,\alpha)}\mathsf{X}_{j;b,n}^{\pm}(0)$, for some $\alpha\in Q$. Thus we get

$$\text{Span of } \{\mathsf{X}_{j;b,n}^{\pm}(\lambda)\}_{\lambda\in\mathfrak{h}^*,b\in S_j^{\pm},n\in\mathbb{N}} = \text{ Span of } \{\mathsf{X}_{j;b,n}^{\pm}(0)\}_{b\in S_j^{\pm},n\in\mathbb{N}}$$

in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{V}_\mu,\mathbb{V}_{\mu\pm\alpha_j})$. This proves the lemma. □

3.8. Throughout this paper, we say $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is *irreducible* if there is no proper non–zero subspace of $\mathbb{V}$ which is stable under $\{\Phi_i(u),\mathfrak{X}_i^{\pm}(u,\lambda)\}_{i\in\mathbf{I}}$. Later, in Corollary 9.3, we will show that this naive notion of irreducibility agrees with the more general one, for any category with an initial object.

Now we are in the position to prove the following

Theorem. *Let $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ be an irreducible, λ–flat object. Then there exists a non–zero, unique up to scalar, weight vector $\Omega\in\mathbb{V}_\mu$ such that*

- $\mathfrak{X}_i^{+}(u,\lambda)\Omega=0$ *for every* $i\in\mathbf{I},u\in\mathbb{C},\lambda\in\mathfrak{h}^*$.
- Ω *is an eigenvector for* $\Phi_i(u)$ *for every* $i\in\mathbf{I}$.
- $\mathbb{V}$ *is the span of the following vectors:*

$$\mathfrak{X}_{i_1}^{-}(u_1,\lambda+\hbar\alpha_{i_1})\cdots\mathfrak{X}_{i_l}^{-}(u_l,\lambda+\hbar(\alpha_{i_1}+\cdots+\alpha_{i_l}))\Omega$$

where $u_1,\cdots,u_l\in\mathbb{C}$, $\lambda\in\mathfrak{h}^*$, $l\in\mathbb{N}$ *and* $i_1,\cdots,i_l\in\mathbf{I}$.

Proof. By the category $\mathcal{O}$ condition, we can find a weight space $\mathbb{V}_\mu$ such that $\mathbb{V}_{\mu+\alpha_i}=0$ for every $i\in\mathbf{I}$. Let $\Omega\in\mathbb{V}_\mu$ be a non–zero eigenvector for $\{\Phi_i(u)\}_{i\in\mathbf{I}}$. Thus there exists $\underline{\mathcal{A}}\in\mathcal{M}(\mu)$ such that $\Phi_i(u)\Omega=A_i(u)\Omega$ for every $i\in\mathbf{I}$. Moreover $\mathfrak{X}_i^{+}(u,\lambda)\Omega=0$ since $\mu+\alpha_i$ is not a weight of $\mathbb{V}$.

Let $W\subset\mathbb{V}$ be the subspace spanned by the following vectors:

$$\mathfrak{X}_{i_1}^{-}(u_1,\lambda+\hbar\alpha_{i_1})\cdots\mathfrak{X}_{i_l}^{-}(u_l,\lambda+\hbar(\alpha_{i_1}+\cdots+\alpha_{i_l}))\Omega$$

over all $u_1,\cdots,u_l\in\mathbb{C}$, $\lambda\in\mathfrak{h}^*$ and $i_1,\cdots,i_l\in\mathbf{I}$. Using the relation ($\mathcal{EQ}3$) and the fact that Ω is an eigenvector for $\Phi_i(u)$ we get that W is stable under $\Phi_i(u)$ for every $i\in\mathbf{I}$.

Now we claim that W is also stable under the raising and lowering operators. Lemma 3.7 implies that we only need to prove this assertion for the operator obtained by specializing the dynamical variable of our choice. This makes it clear that W is stable under $\mathfrak{X}_i^-(u,\lambda)$. For the raising operators, again we have

$$\mathfrak{X}_i^+(u,\lambda')\mathfrak{X}_{i_1}^-(u_1,\lambda+\hbar\alpha_{i_1})\cdots\mathfrak{X}_{i_l}^-(u_l,\lambda+\hbar(\alpha_{i_1}+\cdots+\alpha_{i_l}))\Omega = \sum_{t=1}^{l}\cdots\left[\mathfrak{X}_i^+(u,\lambda'),\mathfrak{X}_{i_t}^-(u_t,\lambda+\hbar(\alpha_{i_1}+\cdots+\alpha_{i_t})\right]\cdots\Omega$$

For λ' such that $\lambda'+\lambda=\hbar(\mu-\alpha_{i_1}-\cdots-\alpha_{i_l})$ we can apply the relation $(\mathcal{EQ}5)$ to each term of the summation on the right–hand side above and conclude that W is stable under $\mathfrak{X}_i^+(u,\lambda')$.

Thus W is a subobject of $\mathbb{V}$. Since $\mathbb{V}$ is assumed to be irreducible, this implies that $W=\mathbb{V}$ as required. □

3.9. **Composition series.** We have the following analogue of the existence of composition series in category $\mathcal{O}$ for a symmetrisable Kac–Moody algebra [34, Lemma 9.6]. We refer the reader to [31, Prop. 15] for a similar statement for quantum loop algebras.

Lemma. *Let $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ be λ–flat and $\mu\in\mathfrak{h}^*$. Then, there exists a filtration*

$$0=\mathbb{V}_0\subset\cdots\subset\mathbb{V}_t=\mathbb{V}$$

such that the following holds for any $\mathbb{V}_j$, $j=1,\ldots,t$

- *either $\mathbb{V}_j/\mathbb{V}_{j-1}$, is an irreducible object with highest weight μ_j for some $\mu_j\geq\mu$,*
- *or $(\mathbb{V}_j/\mathbb{V}_{j-1})_\nu=0$ for every $\nu\geq\mu$.*

Proof. The proof of this lemma is essentially the same as the one given in [34]. Namely, we define

$$a(\mu,\mathbb{V})=\sum_{\nu\geq\mu}\dim(\mathbb{V}_\nu)$$

Note that $a(\mu,\mathbb{V})$ is finite by the category $\mathcal{O}$ axiom from Section 2.3.

We prove the lemma by induction on $a(\mu,\mathbb{V})$. Assuming $a(\mu,\mathbb{V})=0$, the filtration $0\subset\mathbb{V}$ satisfies the conditions of the lemma. Let $a(\mu,\mathbb{V})>0$ and choose a weight $\mu_1\geq\mu$ of $\mathbb{V}$ such that $\mathbb{V}_{\mu+\alpha_i}=0$ for every $i\in\mathbf{I}$. Let $\Omega_1\in\mathbb{V}_{\mu_1}$ be an eigenvector for $\{\Phi_i(u)\}_{i\in\mathbf{I}}$ and consider the subspace W spanned by

$$\mathfrak{X}_{i_1}^-(u_1,\lambda+\hbar\alpha_{i_1})\cdots\mathfrak{X}_{i_l}^-(u_l,\lambda+\hbar(\alpha_{i_1}+\cdots+\alpha_{i_l}))\Omega_1$$

over all $u_1,\cdots,u_l\in\mathbb{C}$, $\lambda\in\mathfrak{h}^*$ and $i_1,\cdots,i_l\in\mathbf{I}$.

The argument given in the proof of Theorem 3.8 yields that W is stable under $\{\Phi_i(u),\mathfrak{X}_i^\pm(u,\lambda)\}$ and hence defines a subobject of $\mathbb{V}$. Note that $\dim(W_{\mu_1})=1$ and for any proper subspace W' of W which is stable under these operators, we must have $\dim(W'_{\mu_1})=0$. Let $W_1\subset W$ be the largest such proper subspace. Then we arrive at a filtration $0\subset W_1\subset W\subset\mathbb{V}$ where W/W_1 is irreducible and $a(\mu,W_1),a(\mu,\mathbb{V}/W)<a(\mu,\mathbb{V})$. Hence we are done by induction.

□

4. Classification of irreducibles I: Necessary condition

In this section we give a necessary condition for the eigenvalues of $\{\Phi_i(u)\}$ on the highest weight vector of an irreducible object in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$.

4.1. Let $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ be an irreducbile, λ–flat object. By Theorem 3.8, we have a unique (up to scalar) weight vector $\Omega \in \mathbb{V}_\mu$ such that

- $\mathfrak{X}_i^+(u,\lambda)\Omega = 0$ for every $i \in \mathbf{I}, u \in \mathbb{C}, \lambda \in \mathfrak{h}^*$.
- There exist $\underline{A} \in \mathcal{M}(\mu)$ such that $\Phi_i(u)\Omega = A_i(u)\Omega$ for every $i \in \mathbf{I}$.
- $\mathbb{V}$ is spanned by the following vectors
$$\mathfrak{X}_{i_1}^-(u_1, \lambda + \hbar\alpha_{i_1}) \cdots \mathfrak{X}_{i_l}^-(u_l, \lambda + \hbar(\alpha_{i_1} + \cdots + \alpha_{i_l}))\Omega$$
over all $u_1, \cdots, u_l \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$ and $i_1, \cdots, i_l \in \mathbf{I}$.

Theorem. *For every $i \in \mathbf{I}$, there exist $\underline{N} = (N_i)_{i\in\mathbf{I}} \in \mathbb{N}^{\mathbf{I}}$ and collection of complex numbers $\{b_1^{(i)}, \cdots, b_{N_i}^{(i)}\}$ such that*
$$A_i(u) = C_i \prod_{k=1}^{N_i} \frac{\theta(u - b_k^{(i)} + d_i\hbar)}{\theta(u - b_k^{(i)})}$$
where C_i is a non–zero constant. In particular $\mu(\alpha_i^\vee) = N_i$ and hence μ is a dominant integral weight.

The proof of this theorem is given in Section 4.10 below.

4.2. We remark that the statement of Theorem 4.1 for Yangians and quantum loop algebras is part of the classification of their irreducible representations by Drinfeld polynomials. For $\mathfrak{g} = \mathfrak{sl}_2$ the classification theorem is proved in [8, 9]. In [11] and [31] the theorem is proved for an arbitrary finite–dimensional semisimple Lie algebra and symmetrisable Kac–Moody algebra respectively (see also [10, Chapter 12], [39, Chapter 3] and references therein).

The proof of Theorem 4.1 given here differs significantly from the ones that exist in the literature due to the lack of standard constructions of PBW basis, Verma modules, tensor product etc. Namely, we produce an infinite sequence of lowering operators which never annihilate the highest–weight vector, if the contrary of Theroem 4.1 were true. Lemma 4.6 is crucial in achieving this. We merely wish to remark that the proofs of this lemma and Theorem 4.1 can be easily modified to work for Yangians and quantum loop algebras.

4.3. **Choice of representatives of poles.** For $i \in \mathbf{I}$ we define $\mathbb{V}^{(i)} = \oplus_{n\geq 0}\mathbb{V}_{\mu - n\alpha_i}$. Note that this is a finite–dimensional vector space which is stable under the action of $\mathfrak{X}_i^\pm(u,\lambda)$ and $\{\Phi_j(u)\}_{j\in\mathbf{I}}$. Let $\sigma(\mathbb{V}, i)$ denote the set of poles of $\{\Phi_i(u), \mathfrak{X}_i^\pm(u,\lambda)\}$ acting on $\mathbb{V}^{(i)}$ in the spectral variable u. We choose a (finite) subset $S_{\mathbb{V}}^{(i)} \subset \mathbb{C}$ satisfying the following conditions

- For every $a \in \sigma(\mathbb{V}, i)$ there is a unique $\overline{a} \in S_{\mathbb{V}}^{(i)}$ such that $a - \overline{a} \in \Lambda_\tau$.
- If $a \in \sigma(\mathbb{V}, i) \cap S_{\mathbb{V}}^{(i)}$ and $a + n\hbar_i \in \sigma(\mathbb{V})$ for some $n \in \mathbb{Z}^\times$ then $a + n\hbar_i \in S_{\mathbb{V}}^{(i)}$.

Here and below we use the notation $\hbar_i = d_i\hbar$.

4.4. **Partial fractions.** Considering the principal parts of the Laurent series expansions of $\Phi_i(u), \mathfrak{X}_i^\pm(u,\lambda)$ near a point $a \in S_{\mathbb{V}}^{(i)}$ we can write the partial fraction decomposition of these functions (see Lemma 3.2) as follows:

$$\mathfrak{X}_i^\pm(u,\lambda) = \sum_{\substack{a\in S_{\mathbb{V}}^{(i)}\\ n\in\mathbb{N}}} \mathsf{X}_{i;a,n}^\pm(\lambda)\frac{(-\partial_u)^n}{n!}\left(\frac{\theta(u-a+\lambda_i)}{\theta(u-a)\theta(\lambda_i)}\right)$$

For each $a \in S_{\mathbb{V}}^{(i)}$ let us define $\mathsf{H}_{i;a,n}$ using the Laurent series expansion of $\Phi_i(u)$ near $u = a$:

$$\Phi_i(u) = \sum_{n\in\mathbb{N}} \frac{\mathsf{H}_{i;a,n}}{(u-a)^{n+1}} + \text{ part holomorphic near } a$$

Then we can rewrite the relations between $\{\Phi_i(u), \mathfrak{X}_i^\pm(u,\lambda)\}$ considered as meromorphic functions taking values in $\mathrm{End}\left(\mathbb{V}^{(i)}\right)$, in the following manner:

Lemma. (1) *For each $a \in S_{\mathbb{V}}^{(i)}$ and $n \in \mathbb{N}$, we have:*

$$\mathrm{Ad}(\Phi_i(u)).\mathsf{X}_{i;a,n}^\pm(\lambda) = \sum_{l\geq 0} \frac{\partial_v^l}{l!}\left(\frac{\theta(u-v\pm\hbar_i)}{\theta(u-v\mp\hbar_i)}\right)\Bigg|_{v=a} \mathsf{X}_{i;a,n+l}^\pm(\lambda\pm\hbar\alpha_i) \tag{4.1}$$

(2) *For each $a \in S_{\mathbb{V}}^{(i)}$ and $m, n \in \mathbb{N}$ we have*

$$\sum_{k,l\geq 0}\frac{(-1)^l}{k!l!}\Big(\theta^{(k+l)}(\mp\hbar_i)\mathsf{X}_{i;a,n+k}^\pm(\lambda)\mathsf{X}_{i;a,m+l}^\pm(\lambda\mp\hbar\alpha_i) - \theta^{(k+l)}(\pm\hbar_i)\mathsf{X}_{i;a,m+l}^\pm(\lambda)\mathsf{X}_{i;a,n+k}^\pm(\lambda\mp\hbar\alpha_i)\Big) = 0 \tag{4.2}$$

(3) *For each $a, b \in S_{\mathbb{V}}^{(i)}$, $m, n \in \mathbb{N}$ we have the following relation on a weight space $\mathbb{V}_\nu$:*

$$\theta(\hbar_i)\left[\mathsf{X}_{i;a,m}^+(\lambda), \mathsf{X}_{i;b,n}^-(-\lambda+\hbar\nu)\right] = \delta_{a,b}\mathsf{H}_{i;a,m+n} \tag{4.3}$$

Proof. (1) was already proved in Lemma 3.3. The proofs of (2) and (3) are along the same lines as that of Lemma 3.3.

(2): Consider the relation ($\mathcal{EQ}4$) or its equivalent form ($\mathcal{EQ}4'$) given in Section 2.3 for $i = j$ and λ replaced by $\lambda - \frac{\hbar}{2}\alpha_i$ (for notational convenience only), multiply it by $(u-a)^n(v-a)^m$. Integrate with respect to both u and v over a small circle centered at a, say $\mathcal{C}$. Note that the second and third terms from both sides vanish and we are left with

$$\begin{aligned}
\text{L.H.S. } &= \oint_{\mathcal{C}}\oint_{\mathcal{C}} (u-a)^n(v-a)^m\theta(u-v\mp\hbar_i)\mathfrak{X}_i^\pm(u,\lambda)\mathfrak{X}_i^\pm(v,\lambda\mp\hbar\alpha_i)\,du\,dv \\
&= \sum_{l\geq 0}\oint_{\mathcal{C}}(u-a)^n\frac{(-1)^l}{l!}\theta^{(l)}(u-a\mp\hbar_i)\mathfrak{X}_i^\pm(u,\lambda)\,du\,\mathsf{X}_{i;a,m+l}^\pm(\lambda\mp\hbar\alpha_i) \\
&= \sum_{k,l\geq 0}\frac{(-1)^l}{k!l!}\theta^{(k+l)}(\mp\hbar_i)\mathsf{X}_{i;a,n+k}^\pm(\lambda)\mathsf{X}_{i;a,m+l}^\pm(\lambda\mp\hbar\alpha_i)
\end{aligned}$$

The same calculation with the right–hand side gives (4.2).

(3): Again we consider ($\mathcal{EQ}5$) with $i=j$ on a weight space $\mathbb{V}_\nu$. Multiply both sides by $(v-b)^n$ and integrate over a small circle $\mathcal{C}$ around b, to get:

$$\theta(\hbar_i)\left[\mathfrak{X}_i^+(u,\lambda),\mathsf{X}^-_{i;b,n}(-\lambda+\hbar\nu)\right]=\oint_{\mathcal{C}}(v-b)^n\frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)}\Phi_i(v)\,dv$$

Now we multiply this by $(u-a)^m$ and integrate around $\mathcal{C}'$, a small circle around a. Clearly the function on the right–hand side of the equation above only has poles at $u=a$ (modulo Λ_τ). Therefore, if $a\neq b\in S^{(i)}_{\mathbb{V}}$, the integral will be zero. Assume $b=a$ and that $\mathcal{C}'$ is a slight enlargement of $\mathcal{C}$. Then we get

$$\begin{aligned}\theta(\hbar_i)\left[\mathsf{X}^+_{i;a,m}(\lambda),\mathsf{X}^-_{i;a,n}(-\lambda+\hbar\nu)\right]&=\oint_{\mathcal{C}}\oint_{\mathcal{C}'}(u-a)^m\frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)}\,du\,(v-a)^n\Phi_i(v)\,dv\\&=\oint_{\mathcal{C}}(v-a)^{n+m}\Phi_i(v)\,dv\\&=\mathsf{H}_{i;a,n+m}\end{aligned}$$

□

4.5. As a consequence of (4.2) we have the following

Corollary. *For every $m,n\in\mathbb{N}$ with $m\geq n$, the following relation holds*

$$\mathsf{X}^\pm_{i;a,m}(\lambda)\mathsf{X}^\pm_{i;a,n}(\lambda\mp\hbar\alpha_i)=\sum_{n\leq r<s}C^\pm_{m,n}(r,s)\mathsf{X}^\pm_{i;a,r}(\lambda)\mathsf{X}^\pm_{i;a,s}(\lambda\mp\hbar\alpha_i)$$

where $C^\pm_{m,n}(r,s)\in\mathbb{C}$.

Proof. The idea of the proof is to use (4.2) and the fact that there exists $N>0$ such that $\mathsf{X}^\pm_{i;a,l}(\lambda)=0$ for every $l>N$. Let us assume that N is smallest such positive integer. Then by (4.2) for $m=n=N$ we have

$$2\theta(\mp\hbar_i)\mathsf{X}^\pm_{i;a,N}(\lambda)\mathsf{X}^\pm_{i;a,N}(\lambda\mp\hbar\alpha_i)=0$$

In order to prove the assertion of the corollary for $m=N$ and $n<N$, we again use (4.2) to get

$$\sum_{k\geq0}\frac{1}{k!}\theta^{(k)}(\mp\hbar)\mathsf{X}^\pm_{i;a,n+k}(\lambda)\mathsf{X}^\pm_{i;a,N}(\lambda\mp\hbar\alpha_i)=\sum_{k\geq0}\frac{1}{k!}\theta^{(k)}(\pm\hbar)\mathsf{X}^\pm_{i;a,N}(\lambda)\mathsf{X}^\pm_{i;a,n+k}(\lambda\mp\hbar\alpha_i)$$

An easy induction argument proves the corollary for $m=N$ and arbitrary n. Now we assume that we have proved it for $m>m_1$ and every $n\leq m$, and continue with proving it for $m=m_1$. Proceeding just as above we have

$$\begin{aligned}\sum_{k,l\geq0}\frac{(-1)^l}{k!l!}\theta^{(k+l)}(\mp\hbar_i)\mathsf{X}^\pm_{i;a,m_1+k}(\lambda)\mathsf{X}^\pm_{i;a,m_1+l}(\lambda\mp\hbar\alpha_i)&\\=\sum_{k,l\geq0}\theta^{(k+l)}(\pm\hbar_i)\mathsf{X}^\pm_{i;a,m_1+l}(\lambda)\mathsf{X}^\pm_{i;a,m_1+k}(\lambda\mp\hbar\alpha_i)=0&\end{aligned}$$

The induction hypothesis implies that the claim holds for $n=m=m_1$. The general case with $n<m_1$ then follows just as before (for the case of $m=N$).

□

4.6. **Main Lemma.**

Lemma. *Let $a \in S_{\mathbb{V}}^{(i)}$ and $N \in \mathbb{N}$. Assume there exists a non–zero weight vector $v \in \mathbb{V}_{\nu}^{(i)}$ satisfying the following two conditions.*

- $\mathsf{X}^{+}_{i;a,k}(\lambda)v = 0$ *for every $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.*
- $\Phi_j(u)v = B_j(u)v$ *for some $\underline{\mathcal{B}} = (B_j(u)) \in \mathcal{M}(\nu)$ such that $B_i(u)$ has a pole of order $N+1$ at a.*

Then we have

$$\mathsf{X}^{-}_{i;a,0}(\lambda)\cdots\mathsf{X}^{-}_{i;a,N}(\lambda + N\hbar\alpha_i)v \neq 0$$

for every $\lambda \in \mathfrak{h}^*$.

Proof. For the purposes of the proof, let us keep $i \in \mathbf{I}$ and $a \in S_{\mathbb{V}}^{(i)}$ fixed and hence omit these from the subscript.

We begin by making the following observation which directly follows from the proofs of Proposition 3.5 and Lemma 3.7:

There exist $\alpha^{(1)}, \alpha^{(2)}, \cdots \in Q$ such that, if we define, $\underline{\mathcal{B}}^{(r)} \in \mathcal{M}(\nu - r\alpha_i)$ by:

$$B_j^{(r)}(u) = B_j(u)\left(\frac{\theta(u-a-d_ia_{ij}\hbar/2)}{\theta(u-a+d_ia_{ij}\hbar/2)}\right)^r \exp\left(-2\pi\iota\hbar\left(\alpha_j, \alpha^{(1)} + \cdots + \alpha^{(r)}\right)\right)$$

Then we have

$$\mathsf{X}^{-}_{i;a,n}(\lambda) = e^{2\pi\iota(\lambda,\alpha^{(r)})}\mathsf{X}^{-}_{i;a,n}(0) : \mathbb{V}_{\nu-(r-1)\alpha_i}[\underline{\mathcal{B}}^{(r-1)}] \to \mathbb{V}_{\nu-r\alpha_i}[\underline{\mathcal{B}}^{(r)}] \tag{4.4}$$

Now define $v(k_1, \cdots, k_r; \lambda) \in \mathbb{V}_{\nu-r\alpha_i}[\underline{\mathcal{B}}^{(r)}]$ by

$$v(k_1, \cdots, k_r; \lambda) := \mathsf{X}^{-}_{i;a,k_1}(\lambda)\cdots\mathsf{X}^{-}_{i;a,k_r}(\lambda + (r-1)\hbar\alpha_i)v$$

Note that by (4.4) above, we have

$$v(k_1, \cdots, k_r; \lambda) = \exp\left(2\pi\iota\left(\lambda, \alpha^{(1)} + \cdots + \alpha^{(r)}\right)\right)v(k_1, \cdots, k_r; 0) \tag{4.5}$$

Let us consider the subspace $\widetilde{W}$ of $\mathbb{V}$ spanned by $v(k_1, \cdots, k_r; \lambda)$ for all $k_1, \cdots, k_r \in \mathbb{N}$ and $\lambda \in \mathfrak{h}^*$ (for $r = 0$, this vector is just v). By the straightening relation given in Corollary 4.5 we have

$$\widetilde{W} = \text{ Span of } \{v(k_1, \cdots, k_r; \lambda)\}_{0 \leq k_1 < \cdots < k_r}$$

Let $W' \subset \widetilde{W}$ be the span of such vectors with $k_r > N$, and let $W = \widetilde{W}/W'$ be the quotient space. Define, for each $n \in \mathbb{N}$ and $0 \leq p \leq n$:

$$v_{p,n}(\lambda) := v(N-n, \cdots, N-n+p-1, \widehat{N-n+p}, N-n+p+1, \cdots, N; \lambda)$$

where as usual $\hat{x}$ means x is skipped. We prove the following assertion by induction on n (for $0 \leq n \leq N$):
$\mathbf{S}_n$: For any fixed $\lambda \in \mathfrak{h}^*$, the collection of vectors $\{v_{p,n}(\lambda)\}_{0 \leq p \leq n}$ is linearly independent in the quotient space W.

Note that the vector $v_{0,n}(\lambda)$ is an eigenvector for $\{\Phi_j(u)\}$ by (4.1) and Corollary 4.5. That is,

$$\Phi_j(u)v_{0,n}(\lambda) = B_j(u)\left(\frac{\theta(u-a-d_ia_{ij}\hbar/2)}{\theta(u-a+d_ia_{ij}\hbar/2)}\right)^n v_{0,n}(\lambda-\hbar\alpha_j) \text{ modulo } W' \quad (4.6)$$

In particular, for each $l \geq 0$ we have a complex number $h_{l,n} \in \mathbb{C}$ such that

$$\mathsf{H}_{i;a,l}v_{0,n}(\lambda) = h_{l,n}v_{0,n}(\lambda-\hbar\alpha_i) \text{ modulo } W' \quad (4.7)$$

The base case $\mathbf{S}_0$ just means that $v \neq 0$ which is true by our hypothesis and the fact that the subspace W' is spanned by weight vectors of weight strictly less than ν (hence $v \notin W'$).

Now we carry out the induction step. Let us first assume $\mathbf{S}_n$ for each $0 \leq n \leq N$. We claim that $v_{0,n+1}(\lambda) \neq 0$. Note that when $n = N$, this is exactly the statement we need to prove. Assume that this vector is zero. Then applying $\mathsf{X}^+_{i;a,0}(-\lambda+\hbar(\nu-(n+1)\alpha_i))$ and using the relation (4.3), we get

$$\sum_{p=0}^{n} \mathsf{X}^-_{i;a,N-n}(\lambda)\cdots\mathsf{X}^-_{i;a,N-n+p-1}(\lambda+(p-1)\hbar\alpha_i)\mathsf{H}_{i;a,N-n+p}$$
$$\mathsf{X}^-_{i;a,N-n+p+1}(\lambda+(p+1)\hbar\alpha_i)\cdots\mathsf{X}^-_{i;a,N}(\lambda+n\hbar\alpha_i)v = 0$$

Note that

$$\mathsf{X}^-_{i;a,N-n+p+1}(\lambda+(p+1)\hbar\alpha_i)\cdots\mathsf{X}^-_{i;a,N}(\lambda+n\hbar\alpha_i)v = v_{0,N-n+p}(\lambda+(p+1)\hbar\alpha_i)$$

and this vector is an eigenvector for H (as given in (4.7)). Thus we get

$$\sum_{p=0}^{n} h_{N-n+p,N-n+p}v_{p,n}(\lambda) = 0$$

This is a non–trivial dependence relation, since $h_{N,N} \neq 0$ by our hypothesis on $B_i(u)$ (it has a pole of order $N+1$ at a), which contradicts $\mathbf{S}_n$.

Assuming $n < N$, we can now prove $\mathbf{S}_{n+1}$. The proof relies on the following easy computation, using (4.1) and Corollary 4.5:

$$\Phi_i(u)v_{p,n}(\lambda) = B_i^{(n)}(u)v_{p,n}(\lambda)+$$
$$B_i^{(n)}(u)\frac{\theta(u-a+\hbar_i)}{\theta(u-a-\hbar_i)}\left(\partial_v\left(\frac{\theta(u-v-\hbar_i)}{\theta(u-v+\hbar_i)}\right)\Bigg|_{v=a}\right)v_{p-1,n}(\lambda)+\ldots$$

where $\ldots$ refers to terms $v_{j,n}$ with $j < p-1$. This triangularity property of $\Phi_i(u)$ together with the fact that $v_{0,n+1} \neq 0$ implies the required result, using the following general fact from linear algebra. □

4.7.

Lemma. *Let V be a vector space over $\mathbb{C}$ and $\{v_1,\cdots,v_l\} \subset V$ such that $v_1 \neq 0$. Assume that there exists $X \in \mathrm{End}(V)$ such that $Xv_i = \sum_{j\leq i} c_{ij}v_j$ where*

- *$c_{ii} = c$ is independent of i.*
- *$c_{i-1,i} \neq 0$ for each i.*

Then $v_1,\cdots,v_l$ are linearly independent.

4.8. Let us note an important consequence of Lemma 4.6.

Define a subspace of $\mathbb{V}$:

$$V' := \text{ Span of } \{\mathsf{X}^-_{i;a,k_0}(\lambda)\cdots\mathsf{X}^-_{i;a,k_N}(\lambda+N\hbar\alpha_i)v\}_{k_0,\cdots,k_N\in\mathbb{N}}$$

This subspace is stable under $\{\Phi_j(u)\}$ because of the relation 4.1. Moreover by Proposition 3.5 we have the following inclusion

$$V' \subset \mathbb{V}^{(i)}_{\nu-(N+1)\alpha_i}\left[\widetilde{\underline{\mathcal{B}}}\right]$$

where $\widetilde{\underline{\mathcal{B}}} = \underline{\mathcal{B}}^{(N+1)}$ in the notation introduced in the proof of Lemma 4.6.

The fact that this subspace is non–zero follows from Lemma 4.6 and hence implies the following

Corollary. *There exists a non–zero vector $v' \in V'$ such that $\Phi_j(u)v' = \widetilde{B}_j(u)v'$ for every $j \in \mathbf{I}$.*

4.9. **Rephrasing Drinfeld polynomials.** For us it will be important to identify meromorphic functions from the location of their zeroes and poles, which are of the following form

$$\mathrm{C}\ \prod_k \frac{\theta(u-c_k+\hbar_i)}{\theta(u-c_k)} \tag{4.8}$$

where $c_k \in \mathbb{C}$ and C is a non–zero constant. A little bit of terminology is in order. A subset $S \subset \mathbb{C}$ is called a $\hbar_i$*–string* if for every $s_1, s_2 \in S$, we have $s_1 - s_2 \in \mathbb{Z}\hbar_i$. Every finite subset of $\mathbb{C}$ can clearly be broken into a finite union of $\hbar_i$–strings.

Given a meromorphic function $B(u)$ such that

$$B(u+1) = B(u) \text{ and } B(u+\tau) = e^{-2\pi\iota\nu_i}B(u)$$

for some $\nu_i \in \mathbb{C}$, let $\sigma(B(u))$ be the set of its zeroes and poles and let $S_B \subset \mathbb{C}$ denote a choice of representatives of the zeroes and poles as outlined in Section 4.3. Let

$$S_B = S_1 \sqcup \ldots \sqcup S_m$$

be the decomposition of S_B into $\hbar_i$–strings. To each string S_k $(1 \le k \le m)$ we associate an expression of parantheses as follows. Let $s \in S_k$ be such that $s + n\hbar_i \notin S_k$ for every $n \geq 1$. If s is a zero (resp. pole) of $B(u)$ of order n, we write n right (resp. left) parantheses. Then we continue with $S_k \setminus \{s\}$ and so on until the $\hbar_i$–string S_k is exhausted. We say that S_k is *balanced* if the resulting expression is of balanced parantheses. The following is immediate.

Lemma. *$B(u)$ is of the form* (4.8) *if, and only if S_k is balanced for each $1 \le k \le m$.*

4.10. **Proof of Theorem 4.1.** Let us assume that there exists $i \in \mathbf{I}$ such that the eigenvalue of $\Phi_i(u)$ on Ω, $A_i(u)$ as introduced in the statement of Theorem 4.1 is not of the form (4.8). Decompose the finite set of representative of zeroes and poles of $A_i(u)$, denoted in the previous section by S_{A_i}, into $\hbar_i$–strings. According to Lemma 4.9, one of these $\hbar_i$–strings is unbalanced. Let S be an unbalanced $\hbar_i$–string in which number of left parantheses is greater than or equal to that of right parantheses (such S must exist since number of zeroes and poles of $A_i(u)$ are the same). We pick $b \in S$ according to the following procedure:

Let E be the expression of parantheses associated to S. By the standard algorithm to determine whether E is balanced or not, one defines a number Counter $= 0$ and reads the expression from right to left. If a paranthesis is right (resp. left) we decrease (resp. increase) the number Counter by 1. Then E is unbalanced if, and only if the Counter becomes positive at some stage (or if upon exhausing E it is non–zero, which will be positive for us, since we are assuming that E has more or equal number of left paratheses). Let x be the paranthesis at which the counter becomes positive for the first time, and let $b \in S$ be the correspoding complex number (necessarily a pole of $A_i(u)$, since left parantheses were associated to poles).

Now we can finish the proof of Theorem 4.1 as follows. Let N_0 be the order of pole of $A_i(u)$ at $b_0 = b$. Applying Lemma 4.6 and Corollary 4.8 to $v = \Omega$, $a = b_0$ and $N = N_0 - 1$ we obtain another (non–zero) vector Ω_1 satisfying the following properties:

- $$\Omega_1 \in \text{Span of } \{\mathsf{X}^-_{i;b_0,k_1}(\lambda)\cdots\mathsf{X}^-_{i;b_0,k_{N_0}}(\lambda+\hbar(N_0-1)\alpha_i)v\}_{k_1,\cdots,k_{N_0}\in\mathbb{N}}$$
- Ω_1 is an eigenvector for $\{\Phi_j(u)\}$ with eigenvalue $\underline{\mathcal{A}}^{(1)}$ where
$$A^{(1)}_j(u) = A_j(u)\left(\frac{\theta(u-b_0-d_ia_{ij}\hbar/2)}{\theta(u-b_0+d_ia_{ij}\hbar/2)}\right)^{N_0}.e^{-2\pi\iota\hbar(\alpha_j,\alpha)}$$
for some $\alpha \in Q$.

The first property in particular implies that $\mathsf{X}^+_{i;c,k}(\lambda)\Omega_1 = 0$ for every $c \neq b_0$, $k \in \mathbb{N}$ and $\lambda \in \mathfrak{h}^*$. Moreover the function $A^{(1)}_i(u)$ has a pole at $b_1 = b_0 - \hbar_i$ of some order, say N_1, since otherwise $A_i(u)$ must have a zero of order $\geq N_0$ at $b_0 - \hbar_i$ thus contradicting our choice of b as outlined above. Hence we can apply Lemma 4.6 and its Corollary 4.8 again to $a = b_1$, $N = N_1 - 1$ and $v = \Omega_1$.

It remains to observe that, according to our choice of b, this procedure will never terminate. Thus we will obtain infinitely many non–zero vectors $\{\Omega_n\}_{n\geq 0}$ which belong to different weight spaces and hence are linearly independent. This contradicts the finite–dimensionality of $\mathbb{V}^{(i)}$ and completes the proof of Theorem 4.1.

4.11. **Knight's lemma.** The following result follows directly from Lemma 3.9, Proposition 3.5 and Theorems 3.8, 4.1.

Proposition. *Let $\mu \in \mathfrak{h}^*$ and $\underline{\mathcal{A}} \in \mathcal{M}(\mu)$. If there exists $\mathbb{V} \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ such that $\mathbb{V}_\mu[\underline{\mathcal{A}}] \neq 0$ then there exists $\overline{\mu} \in P_+$ such that $\mu \leq \overline{\mu}$ and for each $i \in \mathbf{I}$ the function $A_i(u)$ is of the following form:*

$$A_i(u) = C_i \prod_{k=1}^{N_1}\frac{\theta(u-c_{i,k}+d_i\hbar)}{\theta(u-c_{i,k})}\prod_{l=1}^{N_2}\frac{\theta(u-c'_{i,l}-d_i\hbar)}{\theta(u-c'_{i,l})}$$

where $C_i, c_{i,k}, c'_{i,l} \in \mathbb{C}$, and $N_1, N_2 \in \mathbb{N}$ with $N_1 - N_2 = \mu(\alpha_i^\vee)$.

5. Quantum loop algebra

5.1. **The quantum loop algebra $U_q(L\mathfrak{g})$.** Let $q=\exp(\pi\iota\hbar)$. For any $i\in\mathbf{I}$, set $q_i=q^{d_i}$. We use the standard notation for Gaussian integers

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$$

$$[n]_q!=[n]_q[n-1]_q\cdots[1]_q \qquad \left[\begin{array}{c} n \\ k \end{array}\right]_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$$

The quantum loop algebra $U_q(L\mathfrak{g})$ is the $\mathbb{C}$–algebra generated by elements $\{\Psi^{\pm}_{i,\pm r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$, $\{\mathcal{X}^{\pm}_{i,k}\}_{i\in\mathbf{I},k\in\mathbb{Z}}$ and $\{K_h\}_{h\in\mathfrak{h}}$, subject to the following relations

(QL0) For any $i\in\mathbf{I}$

$$\Psi^{\pm}_{i,0}=K_{\pm d_i\alpha_i^\vee}$$

(QL1) For any $i,j\in\mathbf{I}$, $r,s\in\mathbb{N}$ and $h,h'\in\mathfrak{h}$,

$$[\Psi^{\pm}_{i,\pm r},\Psi^{\pm}_{j,\pm s}]=0 \qquad [\Psi^{\pm}_{i,\pm r},\Psi^{\mp}_{j,\mp s}]=0 \qquad [\Psi^{\pm}_{i,\pm r},K_h]=0$$

$$K_hK_{h'}=K_{h+h'} \qquad K_0=1$$

(QL2) For any $i\in\mathbf{I}$, $k\in\mathbb{Z}$ and $h\in\mathfrak{h}$,

$$K_h\mathcal{X}^{\pm}_{i,k}K_h^{-1}=q^{\pm\alpha_i(h)}\mathcal{X}^{\pm}_{i,k}$$

(QL3) For any $i,j\in\mathbf{I}$, $\varepsilon\in\{\pm\}$ and $l\in\mathbb{Z}$

$$\Psi^{\varepsilon}_{i,k+1}\mathcal{X}^{\pm}_{j,l}-q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,l}\Psi^{\varepsilon}_{i,k+1}=q_i^{\pm a_{ij}}\Psi^{\varepsilon}_{i,k}\mathcal{X}^{\pm}_{j,l+1}-\mathcal{X}^{\pm}_{j,l+1}\Psi^{\varepsilon}_{i,k}$$

for any $k\in\mathbb{Z}_{\geq 0}$ if $\varepsilon=+$ and $k\in\mathbb{Z}_{<0}$ if $\varepsilon=-$

(QL4) For any $i,j\in\mathbf{I}$ and $k,l\in\mathbb{Z}$

$$\mathcal{X}^{\pm}_{i,k+1}\mathcal{X}^{\pm}_{j,l}-q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k+1}=q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{i,k}\mathcal{X}^{\pm}_{j,l+1}-\mathcal{X}^{\pm}_{j,l+1}\mathcal{X}^{\pm}_{i,k}$$

(QL5) For any $i,j\in\mathbf{I}$ and $k,l\in\mathbb{Z}$

$$[\mathcal{X}^{+}_{i,k},\mathcal{X}^{-}_{j,l}]=\delta_{ij}\frac{\Psi^{+}_{i,k+l}-\Psi^{-}_{i,k+l}}{q_i-q_i^{-1}}$$

where we set $\Psi^{\pm}_{i,\mp k}=0$ for any $k\geq 1$.

(QL6) For any $i\neq j\in\mathbf{I}$, $m=1-a_{ij}$, $k_1,\dots,k_m\in\mathbb{Z}$ and $l\in\mathbb{Z}$

$$\sum_{\pi\in\mathfrak{S}_m}\sum_{s=0}^{m}(-1)^s\left[\begin{array}{c} m \\ s \end{array}\right]_{q_i}\mathcal{X}^{\pm}_{i,k_{\pi(1)}}\cdots\mathcal{X}^{\pm}_{i,k_{\pi(s)}}\mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k_{\pi(s+1)}}\cdots\mathcal{X}^{\pm}_{i,k_{\pi(m)}}=0$$

5.2. **Category $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$.** We consider the representations of $U_q(L\mathfrak{g})$ whose restriction to $U_q(\mathfrak{g})$ is in category $\mathcal{O}$ and is integrable [31]. Let $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ be the category of such representations. We briefly summarize the results of [30, §2.10–2.13, §3.6]. (1) was proved in [5, 32].

(1) Define

$$\Psi_i(z)^0=\sum_{k\geq 0}\Psi^{-}_{i,-k}z^k \qquad \Psi_i(z)^\infty=\sum_{k\geq 0}\Psi^{+}_{i,k}z^{-k}$$

$$\mathcal{X}^{\pm}_i(z)^0=-\sum_{k\geq 1}\mathcal{X}^{\pm}_{i,-k}z^k \qquad \mathcal{X}^{\pm}_i(z)^\infty=\sum_{k\geq 0}\mathcal{X}^{\pm}_{i,k}z^{-k}$$

Then on a representation $\mathcal{V} \in \mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$, there exist rational $\text{End}(\mathcal{V})$–valued functions $\{\Psi_i(z), \mathcal{X}_i^{\pm}(z)\}$, regular at $z = 0, \infty$ such that $\Psi_i(z)^{0/\infty}$ and $\mathcal{X}_i^{\pm}(z)^{0/\infty}$ are Taylor expansions of $\Psi_i(z)$ and $\mathcal{X}_i^{\pm}(z)$ and $z = 0/\infty$.

(2) Relations (QL1)–(QL5) can be written in terms of these rational functions (see below).

(3) On a representation $\mathcal{V} \in \mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ the relation (QL6) follows from the rest of the relations.

Thus we can describe objects of $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ in the following manner.

Definition. An object of $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ is an $\mathfrak{h}$–diagonalizable module with finite–dimensional weight spaces $\mathcal{V} = \oplus_{\mu \in \mathfrak{h}^*} \mathcal{V}_\mu$ and rational $\text{End}(\mathcal{V})$–valued functions $\{\Psi_i(z), \mathcal{X}_i^{\pm}(z)\}_{i \in \mathbf{I}}$ regular at $z = 0, \infty$ satisfying the following set of axioms:

Category $\mathcal{O}$ and integrability condition.

- There exist $\mu_1, \cdots, \mu_r \in \mathfrak{h}^*$ such that $\mathcal{V}_\mu \neq 0$ implies that $\mu < \mu_k$ for some $k = 1, \cdots, r$.
- For each $\mu \in \mathfrak{h}^*$ such that $\mathcal{V}_\mu \neq 0$ and $i \in \mathbf{I}$, there exists $N > 0$ such that $\mathcal{V}_{\mu - n\alpha_i} = 0$ for all $n \geq N$.

Normalization condition.

- $\Psi_i(\infty) = \Psi_i(0)^{-1} = q_i^{\alpha_i^\vee} =: K_i$, and $\mathcal{X}_i^{\pm}(0) = 0$.

Commutation relations.

($\mathcal{QL}1$) For any $i, j \in \mathbf{I}$, and $h, h' \in \mathfrak{h}$,

$$[\Psi_i(z), \Psi_j(w)] = 0 \qquad [\Psi_i(z), h] = 0$$

($\mathcal{QL}2$) For any $i \in \mathbf{I}$, and $h \in \mathfrak{h}$,

$$[h, \mathcal{X}_i^{\pm}(z)] = \pm\alpha_i(h)\mathcal{X}_i^{\pm}(z)$$

($\mathcal{QL}3$) For any $i, j \in \mathbf{I}$

$$\begin{aligned}&(z - q_i^{\pm a_{ij}} w)\Psi_i(z)\mathcal{X}_j^{\pm}(w)\\ &\qquad = (q_i^{\pm a_{ij}} z - w)\mathcal{X}_j^{\pm}(w)\Psi_i(z) - (q_i^{\pm a_{ij}} - q_i^{\mp a_{ij}})q_i^{\pm a_{ij}} w \mathcal{X}_j^{\pm}(q_i^{\mp a_{ij}} z)\Psi_i(z)\end{aligned}$$

($\mathcal{QL}4$) For any $i, j \in \mathbf{I}$

$$\begin{aligned}&(z - q_i^{\pm a_{ij}} w)\mathcal{X}_i^{\pm}(z)\mathcal{X}_j^{\pm}(w) - (q_i^{\pm a_{ij}} z - w)\mathcal{X}_j^{\pm}(w)\mathcal{X}_i^{\pm}(z)\\ &= z\left(\mathcal{X}_i^{\pm}(\infty)\mathcal{X}_j^{\pm}(w) - q_i^{\pm a_{ij}}\mathcal{X}_j^{\pm}(w)\mathcal{X}_i^{\pm}(\infty)\right) + w\left(\mathcal{X}_j^{\pm}(\infty)\mathcal{X}_i^{\pm}(z) - q_i^{\pm a_{ij}}\mathcal{X}_i^{\pm}(z)\mathcal{X}_j^{\pm}(\infty)\right)\end{aligned}$$

($\mathcal{QL}5$) For any $i, j \in \mathbf{I}$

$$(z - w)[\mathcal{X}_i^{+}(z), \mathcal{X}_j^{-}(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\left(z\Psi_i(w) - w\Psi_i(z) - (z - w)\Psi_i(0)\right)$$

5.3.

Lemma. *Let $\mathcal{V} \in \mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ and let $i \in \mathbf{I}$. Then the set of poles of $\Psi_i(z)$ is contained in the set of poles of $\mathcal{X}_i^{+}(z)$ (resp. $\mathcal{X}_i^{-}(z)$).*

Proof. Consider the relation $(\mathcal{QL}5)$ and its limit as $w \to \infty$. Then we get

$$(q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_i^-(\infty)] = \Psi_i(z) - \Psi_i(0)$$

which proves the $+$ case of the assertion. The $-$ case is proved similarly by taking $z \to \infty$ limit of $(\mathcal{QL}5)$. □

5.4. **Classification of irreducible representations.** The following result was obtained in [11] for the case when $\mathfrak{g}$ is finite–dimensional simple Lie algebra (see also [10, Chaper 12]). For the general case of symmetrisable Kac–Moody algebras, see [31].

Let $\gamma = \{\gamma_{i,\pm m}^{\pm}\}_{i\in\mathbf{I},m\in\mathbb{N}}$ be a collection of complex numbers and $\mu \in \mathfrak{h}^*$ such that $\gamma_{i,0}^{\pm} = q_i^{\pm\mu(\alpha_i^\vee)}$. A representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ is said to be an l–highest weight representation of l–highest weight (μ, γ) if there exists $\mathbf{v} \in \mathcal{V}$ such that

(1) $\mathcal{V} = U_q(L\mathfrak{g})\mathbf{v}$.
(2) $\mathcal{X}_{i,k}^+\mathbf{v} = 0$ for every $i \in \mathbf{I}$ and $k \in \mathbb{Z}$.
(3) $\Psi_{i,\pm m}^{\pm}\mathbf{v} = \gamma_{i,\pm m}^{\pm}\mathbf{v}$ and $K_h\mathbf{v} = q^{\mu(h)}\mathbf{v}$ for any $i \in \mathbf{I}, m \in \mathbb{N}$ and $h \in \mathfrak{h}$.

For any (μ, γ), there is a unique irreducible representation with l–highest weight (μ, γ).

Theorem.

(1) *Every irreducible representation in $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ is a highest weight representation for a unique highest weight (μ, γ).*
(2) *The irreducible representation $\mathcal{V}(\mu, \gamma)$ is in $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ if, and only if there exist monic polynomials $\{\mathcal{P}_i(w) \in \mathbb{C}[w]\}_{i\in\mathbf{I}}$, $\mathcal{P}_i(0) \neq 0$, such that*

$$\sum_{m\geq 0} \gamma_{i,m}^+ z^{-m} = q_i^{-\deg(\mathcal{P}_i)} \frac{\mathcal{P}_i(q^2 z)}{\mathcal{P}_i(z)} = \sum_{m\leq 0} \gamma_{i,m}^- z^m$$

Note that we have $\mu(\alpha_i^\vee) = \deg(\mathcal{P}_i)$ and hence μ is a dominant integral weight. The polynomials $\mathcal{P}_i$ are called Drinfeld polynomials. The set of isomorphism classes of simple objects in $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ is in bijection with the set $\mathcal{P}_+^U$ of pairs $(\mu \in \mathfrak{h}^*, \{\mathcal{P}_i\})$ such that $\mathcal{P}_i$ are monic, $\mathcal{P}_i(0) \neq 0$ and $\mu(\alpha_i^\vee) = \deg(\mathcal{P}_i)$. We denote by $\mathcal{V}(\mu, \{\mathcal{P}_i\})$ the irreducible $U_q(L\mathfrak{g})$–module corresponding to $(\mu, \{\mathcal{P}_i\}) \in \mathcal{P}_+^U$.

6. Construction of the functor

The aim of this section is to construct a functor Θ from a dense subcategory of $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ to $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$.

6.1. **Non–congruent representations.** A representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ is said to be non–congruent if for any $\alpha \neq \beta$ poles of $\mathcal{X}_i^+(z)$ (resp. $\mathcal{X}_i^-(z)$) $\alpha\beta^{-1} \notin p^{\mathbb{Z}}$. Let $\mathcal{O}_{\text{int}}^{\text{NC}}(U_q(L\mathfrak{g}))$ be the full subcategory of $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ consisting of non–congruent representations.

6.2. **Contours.** By a Jordan curve $\mathcal{C}$ we shall mean a disjoint union of simple closed curves in $\mathbb{C}$, the inner domains of which are pairwise disjoint. For $\mathcal{C}$ a Jordan curve and f a continuous function on $\mathcal{C}$, we set

$$\oint_{\mathcal{C}} f(u)\,du = \frac{1}{2\pi\iota}\int_{\mathcal{C}} f(u)\,du$$

The definition of the functor Θ relies upon the following choice of contours of integration. For $\mathcal{V} \in \mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$, $\mu \in \mathfrak{h}^*$ a weight of $\mathcal{V}$ and $i \in \mathbf{I}$, we choose a contour $\mathcal{C}_{i,\mu}^{\pm}$ such that

- $\mathcal{C}_{i,\mu}^{\pm}$ encloses a representative (modulo $\mathbb{Z}$) of each pole of $\mathcal{X}_i^{\pm}(\exp(2\pi\iota u))$ acting on weight space $\mathcal{V}_\mu$.
- $\mathcal{C}_{i,\mu}^{\pm}$ does not enclose any $\mathbb{Z}_{\neq 0}\tau$ translates of the poles of $\mathcal{X}_i^{\pm}(\exp(2\pi\iota u))$ acting on weight space $\mathcal{V}_\mu$ and $\mathcal{V}_{\mu\pm\alpha_i}$.

6.3. **Main construction.** Given $\mathcal{V} \in \mathcal{O}_{\text{int}}^{\text{NC}}(U_q(L\mathfrak{g}))$ we set $\Theta(\mathcal{V}) = \mathcal{V}$ as a diagonalizable $\mathfrak{h}$–module. For each weight $\mu \in \mathfrak{h}^*$ of $\mathcal{V}$ and $i \in \mathbf{I}$ we define operators $\Phi_i(u), \mathfrak{X}_i^{\pm}(u,\lambda)$ on $\mathcal{V}_\mu$ as follows:

(1) Define

$$G_i^{\pm}(z) := \prod_{n\geq 1}(K_i^{\pm 1}\Psi_i(p^{\pm n}z)) \tag{6.1}$$

Note that $G_i^{\pm}(z)$ satisfy the following multiplicative difference equations

$$G_i^+(pz) = [K_i\Psi_i(pz)]^{-1}\,G_i^+(z) \qquad \text{and} \qquad G_i^-(pz) = \left[K_i^{-1}\Psi_i(z)\right]G_i^-(z)$$

Since $K_i\Psi_i(0) = K_i^{-1}\Psi_i(\infty) = 1$, these equations are regular near $z = 0, \infty$ respectively. Therefore, it follows (see *e.g.,* [41]) that $G_i^{\pm}(z)$ are holomorphic in a neighborhood of $z = 0, \infty$ respectively and $G_i^+(0) = G_i^-(\infty) = 1$.

$$\Phi_i(u) = G_i^+(z)\Psi_i(z)G_i^-(z)\big|_{z=\exp(2\pi\iota u)} \tag{6.2}$$

Hence, by construction $\Phi_i(u)$ is 1–periodic and we have

$$\Phi_i(u+\tau) = K_i^{-2}\Phi_i(u) = e^{-2\pi\iota\hbar d_i\alpha_i^\vee}\Phi_i(u)$$

(2) Define

$$\mathfrak{X}_i^{\pm}(u,\lambda) = c_i^{\pm}\oint_{\mathcal{C}_{i,\mu}^{\pm}} \frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)} G_i^{\pm}(\exp(2\pi\iota v))\mathcal{X}_i^{\pm}(\exp(2\pi\iota v))\,dv \tag{6.3}$$

where $\lambda_i = (\lambda, \alpha_i)$. And we consider the right–hand side as defining a holomorphic function outside of the shifts of the contour $\mathcal{C}_{i,\mu}^{\pm}$ by elements of the lattice Λ_τ. Moreover, $c_i^{\pm}$ are constants which are to be chosen to satisfy the following

$$c_i^+c_i^- = (2\pi\iota)^2\frac{\theta^+(0)^2}{\theta^+(d_i\hbar)^2} \tag{6.4}$$

Remark. The operators defined by (6.3) are independent of the choice of the contours $\mathcal{C}_{i,\mu}^{\pm}$ satisfying the conditions of §6.2. This is because, by (6.1) and Lemma 5.3, the poles of $G_i^{\pm}(z)$ are contained in $p^{\mathbb{Z}_{\leqslant 0}}$–multiples of the poles of $\mathcal{X}_i^{\pm}(z)$. In particular, $G_i^{\pm}(\exp(2\pi\iota u))$ are holomorphic within $\mathcal{C}_{i,\mu}^{\pm}$.

Theorem. *The operators constructed above satisfy the axioms of §2.3. Hence we obtain a functor* $\Theta : \mathcal{O}^{\mathrm{NC}}_{\mathrm{int}}(U_q(L\mathfrak{g})) \to \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ *which is faithful and exact.*

6.4. Let $i, j \in \mathbf{I}$ and set $a = d_i a_{ij}\hbar/2$. Consider a contour $\mathcal{C}$ with interior domain D and $\Omega_1, \Omega_2 \subset \mathbb{C}$ two open subsets with $\overline{D} \subset \Omega_2$. Assume given a holomorphic function $f(u,v) : \Omega_1 \times \Omega_2 \to \mathrm{End}(\mathcal{V})$ such that $[\Psi_i(\exp(2\pi\iota u)), f(u,v)] = 0$ for any u, v. We have the following analog of [30, Proposition 5.5].

Proposition. *For each* $\epsilon \in \{\pm 1\}$ *we have:*

(1) *If* $u \notin \overline{D} \pm \epsilon a + \mathbb{Z}$*, then*

$$\mathrm{Ad}(\Psi_i\left(e^{2\pi\iota u}\right))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv = \oint_{\mathcal{C}} \left(\frac{e^{2\pi\iota(u+\epsilon a)} - e^{2\pi\iota v}}{e^{2\pi\iota u} - e^{2\pi\iota(v+\epsilon a)}}\right)^{\pm 1} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv$$

(2) *For* $u \notin \overline{D} + \mathbb{Z} \pm \epsilon a + \mathbb{Z}_{<0}\tau$*, we have*

$$\mathrm{Ad}(G_i^{+}\left(e^{2\pi\iota u}\right))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv = \oint_{\mathcal{C}} \left(\frac{\theta^{+}(u-v+\epsilon a)}{\theta^{+}(u-v-\epsilon a)}\right)^{\pm 1} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv$$

(3) *For* $u \notin \overline{D} + \mathbb{Z} \pm \epsilon a + \mathbb{Z}_{>0}\tau$*, we have*

$$\mathrm{Ad}(G_i^{-}\left(e^{2\pi\iota u}\right))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv = \oint_{\mathcal{C}} \left(\frac{\theta^{-}(u-v+\epsilon a)}{\theta^{-}(u-v-\epsilon a)}\right)^{\pm 1} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv$$

(4) *For* $u \notin \overline{D} \pm \epsilon a + \Lambda_\tau$*, we have*

$$\mathrm{Ad}(\Phi_i(u))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv = \oint_{\mathcal{C}} \left(\frac{\theta(u-v+\epsilon a)}{\theta(u-v-\epsilon a)}\right)^{\pm 1} f(u,v)\mathcal{X}_j^{\epsilon}\left(e^{2\pi\iota v}\right)\,dv$$

6.5. **Proof of ($\mathcal{EQ}$3).** Let $i, j \in \mathbf{I}$ and let $a = d_i a_{ij}\hbar/2$. For $\lambda \in \mathfrak{h}^*$, we write $\lambda_j = (\lambda, \alpha_j)$.

Using the definition of $\mathcal{X}_j^{\pm}(v,\lambda)$ given in (6.3) and Proposition 6.4, the left–hand side of ($\mathcal{EQ}$3) is given by

$$c_j^{\pm} \oint_{\mathcal{C}_j^{\pm}} \frac{\theta(u-v'\pm a)}{\theta(u-v'\mp a)} \frac{\theta(v-v'+\lambda_j)}{\theta(v-v')\theta(\lambda_j)} G_j^{\pm}\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^{\pm}\left(e^{2\pi\iota v'}\right)\,dv'$$

and by definition the right–hand side of ($\mathcal{EQ}$3) is given by a similar contour integral

$$c_j^{\pm} \oint_{\mathcal{C}_j^{\pm}} \mathcal{K}(u,v,v') G_j^{\pm}\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^{\pm}\left(e^{2\pi\iota v'}\right)\,dv'$$

where

$$\mathcal{K} = \frac{\theta(u-v\pm a)\theta(v-v'+\lambda_j\pm 2a)}{\theta(u-v\mp a)\theta(\lambda_j\pm 2a)\theta(v-v')} + \frac{\theta(\pm 2a)\theta(u-v\mp a-\lambda_j)\theta(u-v'+\lambda_j\pm a)}{\theta(\lambda_j)\theta(u-v\mp a)\theta(\lambda_j\pm 2a)\theta(u-v'\mp a)}$$

Thus we have to prove the following identity

$$\frac{\theta(u-v'\pm a)}{\theta(u-v'\mp a)} \frac{\theta(v-v'+\lambda_j)}{\theta(v-v')\theta(\lambda_j)} = \frac{\theta(u-v\pm a)\theta(v-v'+\lambda_j\pm 2a)}{\theta(u-v\mp a)\theta(\lambda_j\pm 2a)\theta(v-v')} + \frac{\theta(\pm 2a)\theta(u-v\mp a-\lambda_j)\theta(u-v'+\lambda_j\pm a)}{\theta(\lambda_j)\theta(u-v\mp a)\theta(\lambda_j\pm 2a)\theta(u-v'\mp a)}$$

Clearing the denominator, the required identity takes the following form (where for notational convenience, we write $b = \pm a$).

$$\theta(u - v' + b)\theta(v - v' + \lambda_j)\theta(u - v - b)\theta(\lambda_j + 2b) =$$
$$\theta(u-v+b)\theta(v-v'+\lambda_j+2b)\theta(u-v'-b)\theta(\lambda_j)+\theta(2b)\theta(u-v-\lambda_j-b)\theta(u-v'+\lambda_j+b)\theta(v-v')$$

which is precisely (FTI).

6.6. **Proof of ($\mathcal{EQ}4$).** Let $i, j \in \mathbf{I}$ and let $\lambda \in \mathfrak{h}^*$. Let us write $b = \pm a$ where $a = d_i a_{ij}\hbar/2$ and $\lambda_k = (\lambda, \alpha_k)$ for $k = i, j$. With these notations in mind, the left–hand side of ($\mathcal{EQ}4$) can be written as

$$\text{L.H.S.} = c_i^\pm c_j^\pm \oint_{\mathcal{C}_j^\pm}\oint_{\mathcal{C}_i^\pm} I(b) G_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right) G_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right) du'\, dv'$$

where

$$\begin{aligned} I(b) = &\frac{\theta(\lambda_i+\lambda_j)\theta(u-v-b)\theta(u-u'+\lambda_i+b)\theta(v-v'+\lambda_j-b)}{\theta(u-u')\theta(\lambda_i+b)\theta(v-v')\theta(\lambda_j-b)} \\ &- \frac{\theta(u-v-\lambda_j)\theta(u-u'+\lambda_i+b)\theta(u-v'+\lambda_i+\lambda_j-b)}{\theta(u-u')\theta(u+\lambda_i-v')\theta(\lambda_j-b)} \\ &- \frac{\theta(u-v+\lambda_i)\theta(v-u'+\lambda_i+\lambda_j+b)\theta(v-v'+\lambda_j-b)}{\theta(v+\lambda_j-u')\theta(v-v')\theta(\lambda_i+b)} \end{aligned}$$

Similarly, the right–hand side of ($\mathcal{EQ}4$) is equal to

$$\text{R.H.S.} = c_i^\pm c_j^\pm \oint_{\mathcal{C}_j^\pm}\oint_{\mathcal{C}_i^\pm} I(-b) G_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right) G_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right) du'\, dv'$$

Claim: We have the following identity. In particular, $I(b) = 0$ for $u' - v' - b \in \Lambda_\tau$.

$$\theta(u' - v' + b)I(b) = \theta(u' - v' - b)I(-b)$$

The proof of this claim is given in the next section. Assuming this, we can finish the proof of ($\mathcal{EQ}4$) as follows. Using Proposition 6.4 we have

$$\begin{aligned} &\oint_{\mathcal{C}} I(b) G_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right) du' \cdot G_j^\pm\left(e^{2\pi\iota v'}\right) = \\ &\qquad G_j^\pm\left(e^{2\pi\iota v'}\right)\oint_{\mathcal{C}} I(b)\frac{\theta^\pm(v'-u'-b)}{\theta^\pm(v'-u'+b)} G_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right) du' \end{aligned} \tag{6.5}$$

as long as v' does not lie in $\overline{D} + \mathbb{Z} \mp a \mp \mathbb{Z}_{>0}\tau$. However, using the claim above, the integrand on the right–hand side is regular on $\mathcal{C} + \mathbb{Z} \mp a \mp \mathbb{Z}_{>0}\tau$. This is because the simple poles of $1/\theta^\pm(v' - u' + b)$ are cancelled by the zeroes of $I(b)$.

Thus multiplying (6.5) by $\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right)$ and integrating v' over the contour $\mathcal{C}_j^\pm$, we obtain the following expression for the left–hand side of ($\mathcal{EQ}4$) divided by $c_i^\pm c_j^\pm$.

$$\oint_{\mathcal{C}_j^\pm}\oint_{\mathcal{C}_i^\pm} I(b)\frac{\theta^\pm(v'-u'-b)}{\theta^\pm(v'-u'+b)} G_i^\pm\left(e^{2\pi\iota u'}\right) G_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right) du'\, dv'$$

Similarly, the right–hand side of ($\mathcal{EQ}4$) divided by $c_i^\pm c_j^\pm$ is equal to

$$\oint_{\mathcal{C}_j^\pm}\oint_{\mathcal{C}_i^\pm} I(-b)\frac{\theta^\pm(u'-v'-b)}{\theta^\pm(u'-v'+b)}G_i^\pm\left(e^{2\pi\iota u'}\right)G_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right)du'\,dv'$$

Using ($\mathcal{QL}4$) this expression can be rewritten as

$$\oint_{\mathcal{C}_j^\pm}\oint_{\mathcal{C}_i^\pm} I(-b)\frac{\theta^\pm(u'-v'-b)}{\theta^\pm(u'-v'+b)}$$
$$\cdot\frac{e^{2\pi\iota(v'+b)}-e^{2\pi\iota u'}}{e^{2\pi\iota v'}-e^{2\pi\iota(u'+b)}}G_i^\pm\left(e^{2\pi\iota u'}\right)G_j^\pm\left(e^{2\pi\iota v'}\right)\mathcal{X}_i^\pm\left(e^{2\pi\iota u'}\right)\mathcal{X}_j^\pm\left(e^{2\pi\iota v'}\right)du'\,dv'$$

Thus, it suffices to check that

$$I(b)\frac{\theta^\pm(v'-u'-b)}{\theta^\pm(v'-u'+b)}=I(-b)\frac{\theta^\pm(u'-v'-b)}{\theta^\pm(u'-v'+b)}\frac{e^{2\pi\iota(v'+b)}-e^{2\pi\iota u'}}{e^{2\pi\iota v'}-e^{2\pi\iota(u'+b)}}$$

Using the claim, this equation is equivalent to the following

$$\theta(u'-v'-b)\frac{\theta^\pm(v'-u'-b)}{\theta^\pm(v'-u'+b)}=\theta(u'-v'+b)\frac{\theta^\pm(u'-v'-b)}{\theta^\pm(u'-v'+b)}\frac{e^{2\pi\iota(v'+b)}-e^{2\pi\iota u'}}{e^{2\pi\iota v'}-e^{2\pi\iota(u'+b)}}$$

which follows directly from (2.3).

6.7. **Proof of the claim.** Recall that $I(b)=T_1(b)-T_2(b)-T_3(b)$ where we have

$$\begin{aligned}
T_1(b)&=\frac{\theta(\lambda_i+\lambda_j)\theta(u-v-b)\theta(u-u'+\lambda_i+b)\theta(v-v'+\lambda_j-b)}{\theta(u-u')\theta(\lambda_i+b)\theta(v-v')\theta(\lambda_j-b)}\\
T_2(b)&=\frac{\theta(u-v-\lambda_j)\theta(u-u'+\lambda_i+b)\theta(u-v'+\lambda_i+\lambda_j-b)}{\theta(u-u')\theta(u+\lambda_i-v')\theta(\lambda_j-b)}\\
T_3(b)&=\frac{\theta(u-v+\lambda_i)\theta(v-u'+\lambda_i+\lambda_j+b)\theta(v-v'+\lambda_j-b)}{\theta(v+\lambda_j-u')\theta(v-v')\theta(\lambda_i+b)}
\end{aligned}$$

Using (FTI) we can easily verify that

$$\begin{aligned}
\theta(u'-v'+b)T_2(b)-\theta(u'-v'-b)T_2(-b)&=\frac{\theta(2b)\theta(u-v-\lambda_j)\theta(u'-v'+\lambda_j)\theta(u-u'+\lambda_i+\lambda_j)}{\theta(u-u')\theta(\lambda_j-b)\theta(\lambda_j+b)}\\
\theta(u'-v'+b)T_3(b)-\theta(u'-v'-b)T_3(-b)&=-\frac{\theta(2b)\theta(u-v+\lambda_i)\theta(u'-v'-\lambda_i)\theta(v-v'+\lambda_i+\lambda_j)}{\theta(v-v')\theta(\lambda_i-b)\theta(\lambda_i+b)}
\end{aligned}$$

Using these, we get the following:

$$\begin{aligned}
&\theta(u-u')\theta(v-v')\theta(\lambda_i-b)\theta(\lambda_i+b)\theta(\lambda_j-b)\theta(\lambda_j+b)(\theta(u'-v'+b)I(b)-(u'-v'-b)I(-b))=\\
&\theta(u-v-b)\theta(u'-v'+b)\theta(u-u'+\lambda_i+b)\theta(v-v'+\lambda_j-b)\theta(\lambda_i+\lambda_j)\theta(\lambda_i-b)\theta(\lambda_j+b)\\
&-\theta(u-v+b)\theta(u'-v'-b)\theta(u-u'+\lambda_i-b)\theta(v-v'+\lambda_j+b)\theta(\lambda_i+\lambda_j)\theta(\lambda_i+b)\theta(\lambda_j-b)\\
&-\theta(u-v-\lambda_j)\theta(u'-v'+\lambda_j)\theta(u-u'+\lambda_i+\lambda_j)\theta(u-u')\theta(\lambda_j+b)\theta(\lambda_j-b)\theta(2b)\\
&+\theta(u-v+\lambda_i)\theta(u'-v'-\lambda_i)\theta(v-v'+\lambda_i+\lambda_j)\theta(v-v')\theta(\lambda_i+b)\theta(\lambda_i-b)\theta(2b)
\end{aligned}$$

We need to prove that the right–hand side of the equation written above is zero. The following is a standard argument involving elliptic functions. Let us keep all the variables, except for u fixed and define:

$$\begin{aligned}F_1(u) &= \theta(u-v-b)\theta(u'-v'+b)\theta(u-u'+\lambda_i+b)\theta(v-v'+\lambda_j-b)\theta(\lambda_i+\lambda_j)\theta(\lambda_i-b)\theta(\lambda_j+b)\\ &-\theta(u-v-\lambda_j)\theta(u'-v'+\lambda_j)\theta(u-u'+\lambda_i+\lambda_j)\theta(u-u')\theta(\lambda_j+b)\theta(\lambda_j-b)\theta(2b)\\ &+\theta(u-v+\lambda_i)\theta(u'-v'-\lambda_i)\theta(v-v'+\lambda_i+\lambda_j)\theta(v-v')\theta(\lambda_i+b)\theta(\lambda_i-b)\theta(2b)\end{aligned}$$

$$F_2(u) = \theta(u-v+b)\theta(u'-v'-b)\theta(u-u'+\lambda_i-b)\theta(v-v'+\lambda_j+b)\theta(\lambda_i+\lambda_j)\theta(\lambda_i+b)\theta(\lambda_j-b)$$

Note that both these functions have the same quasi–periodicity properties:

$$F_s(u+1) = -F_s(u) \qquad \text{and} \qquad F_s(u+\tau) = -e^{-3\pi\tau}e^{-2\pi\iota(3u-v-u'-v'+2\lambda_i)}F_s(u)$$

Also $F_2(u)$ has zeroes at $u = v-b$ and $u = u'-\lambda_i+b$ (modulo Λ_τ). The fact that $F_1(v-b) = 0$ and $F_1(u'-\lambda_i+b) = 0$ is an easy application of (FTI). Thus we deduce that F_1/F_2 is a holomorphic doubly–periodic function of u implying that it is a constant, say C (constant here means independent of u). To get that constant we use (FTI) again to conclude that $F_1(v+b) = F_2(v+b)$, hence this constant must be 1. This proves that $F_1 = F_2$ which implies the desired claim.

6.8. **Proof of ($\mathcal{EQ}$5).** Let $i, j \in \mathbf{I}$ and $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ such that $\lambda_1+\lambda_2 = \hbar(\mu+\alpha_i-\alpha_j)$. Let us write $c = (c_i^+ c_j^-)^{-1}$, so that

$$\begin{aligned}c[\mathfrak{X}_i^+(u,\lambda_1), \mathfrak{X}_j^-(v,\lambda_2)] &= \oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+} \mathcal{T} G_i^+\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)G_j^-\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)du'\,dv'\\ &-\oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+} \mathcal{T} G_j^-\left(e^{2\pi\iota v'}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)G_i^+\left(e^{2\pi\iota u'}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)du'\,dv'\end{aligned}$$

where

$$\mathcal{T} = \frac{\theta(u-u'+\lambda_{1,i})}{\theta(u-u')\theta(\lambda_{1,i})}\frac{\theta(v-v'+\lambda_{2,j})}{\theta(v-v')\theta(\lambda_{2,j})}$$

We would like to apply Proposition 6.4 in order to permute the factors $\mathcal{X}_i^+$ and G_j^- in the first term, X_j^- and G_i^+ in the second. This cannot be done directly however, since $v' \in \mathcal{C}_j^-$ may lie inside $\overline{D_i}^+ - a + \mathbb{Z} + \mathbb{Z}_{>0}\tau$ (and similarly for u'). This problem was encountered in [30, §5.9] and we employ the same method as in there to circumvent this. Namely, let $\delta \in \mathbb{C}$ and define

$$\begin{aligned}J(\delta) &= \oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+} \mathcal{T} G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)G_j^-\left(e^{2\pi\iota(v'-\delta)}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)du'\,dv'\\ &-\oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+} \mathcal{T} G_j^-\left(e^{2\pi\iota(v'-\delta)}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)du'\,dv'\end{aligned}$$

We consider this integral for δ in a disc $|\delta| < R$, where R is such that G_i^+ and G_j^- are holomorphic in $\overline{D_i}^+ + \delta'$ and $\overline{D_j}^- - \delta'$ for any δ' with $|\delta'| < R$.

Moreover, if the contours $\mathcal{C}_i^+$ and $\mathcal{C}_j^-$ are small enough, there is an $r < R$ such that if $|\delta| > r$, then $\overline{D_j}^- - \delta$ is disjoint from $\overline{D_i}^+ - a + \mathbb{Z} + \mathbb{Z}_{>0}\tau$, and $\overline{D_i}^+ + \delta$ is disjoint from $\overline{D_j}^{-} + a + \mathbb{Z} + \mathbb{Z}_{<0}\tau$.

Assuming $r < |\delta| < R$, we can apply Proposition 6.4 to find that $J(\delta)$ equals

$$\oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+}\mathcal{T}\frac{\theta^+(u'-v'+a+\delta)}{\theta^+(u'-v'-a+\delta)}G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)G_j^-\left(e^{2\pi\iota(v'-\delta)}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)du'\,dv'$$
$$-\oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+}\mathcal{T}\frac{\theta^+(u'-v'+a+\delta)}{\theta^+(u'-v'-a+\delta)}G_j^-\left(e^{2\pi\iota(v'-\delta)}\right)G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)\mathcal{X}_j^-\left(e^{2\pi\iota v'}\right)\mathcal{X}_i^+\left(e^{2\pi\iota u'}\right)du'\,dv'$$

where we have used that $\theta^+(u) = \theta^-(-u)$. Using ($\mathcal{QL}5$) we get that the right–hand side is zero for $i \neq j$. From now on we assume that $i = j$ and hence $a = d_i\hbar$. Then we get

$$(q_i - q_i^{-1})J(\delta) = \oint_{\mathcal{C}_j^-}\oint_{\mathcal{C}_i^+}\mathcal{T}\frac{\theta^+(u'-v'+d_i\hbar+\delta)}{\theta^+(u'-v'-d_\hbar+\delta)}G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)G_i^-\left(e^{2\pi\iota(v'-\delta)}\right)$$
$$\cdot\frac{e^{2\pi\iota u'}\Psi_i\left(e^{2\pi\iota v'}\right) - e^{2\pi\iota v'}\Psi_i\left(e^{2\pi\iota u'}\right)}{e^{2\pi\iota u'} - e^{2\pi\iota v'}}\,du'\,dv'$$

Writing

$$\frac{e^{2\pi\iota u'}\Psi_i\left(e^{2\pi\iota v'}\right) - e^{2\pi\iota v'}\Psi_i\left(e^{2\pi\iota u'}\right)}{e^{2\pi\iota u'} - e^{2\pi\iota v'}} =$$
$$\frac{e^{-2\pi\iota v'}\Psi_i\left(e^{2\pi\iota v'}\right) - e^{-2\pi\iota u'}\Psi_i\left(e^{2\pi\iota u'}\right)}{u'-v'}\cdot\frac{u'-v'}{e^{-2\pi\iota v'} - e^{-2\pi\iota u'}}$$

and taking the two contours to be equal, say $\mathcal{C}$, we can apply [30, Lemma 5.8] to conclude that

$$(q_i - q_i^{-1})J(\delta) = \frac{1}{2\pi\iota}\oint_{\mathcal{C}}\overline{\mathcal{T}}\frac{\theta^+(\delta+d_i\hbar)}{\theta^+(\delta-d_i\hbar)}G_i^+\left(e^{2\pi\iota(u'+\delta)}\right)G_i^-\left(e^{2\pi\iota(u'-\delta)}\right)\Psi_i\left(e^{2\pi\iota u'}\right)du' \tag{6.6}$$

where $\overline{\mathcal{T}} = \mathcal{T}|_{v'=u'}$, that is,

$$\overline{\mathcal{T}} = \frac{\theta(u-u'+\lambda_{1,i})}{\theta(u-u')\theta(\lambda_{1,i})}\frac{\theta(v-u'+\lambda_{2,j})}{\theta(v-u')\theta(\lambda_{2,j})}$$

Now both sides of (6.6) are defined for $|\delta| < R$. Thus we conclude that $J(0)$ is given by

$$\begin{aligned}(q_i - q_i^{-1})J(0) &= \frac{1}{2\pi\iota}\frac{\theta^+(d_i\hbar)}{\theta^-(d_i\hbar)}\oint_{\mathcal{C}}\overline{\mathcal{T}}G_i^+\left(e^{2\pi\iota u'}\right)G_i^-\left(e^{2\pi\iota u'}\right)\Psi_i\left(e^{2\pi\iota u'}\right)du'\\ &= \frac{1}{2\pi\iota}\frac{\theta^+(d_i\hbar)}{\theta^-(d_i\hbar)}\oint_{\mathcal{C}}\overline{\mathcal{T}}\Phi_i(u')\,du'\end{aligned}$$

Thus, we have the following

$$\begin{aligned}\theta(d_i\hbar)[\mathfrak{X}_i^+(u,\lambda_1),\mathfrak{X}_i^-(v,\lambda_2)] &= \frac{c_i^+c_i^-}{2\pi\iota}\frac{\theta^+(d_i\hbar)}{\theta^-(d_i\hbar)}\frac{\theta(d_i\hbar)}{q_i-q_i^{-1}}\oint_{\mathcal{C}}\overline{\mathcal{T}}\Phi_i(u')\,du' \\ &= \frac{(2\pi\iota)^2}{2\pi\iota}\frac{\theta^+(0)^2}{\theta^+(d_i\hbar)^2}\frac{\theta^+(d_i\hbar)}{\theta^-(d_i\hbar)}\frac{\theta(d_i\hbar)}{q_i-q_i^{-1}}\oint_{\mathcal{C}}\overline{\mathcal{T}}\Phi_i(u')\,du' \\ &= \oint_{\mathcal{C}}\overline{\mathcal{T}}\Phi_i(u')\,du'\end{aligned}$$

where we use the equation (6.4) in the second equation and (2.3) in the third.

In order to carry out the computation, we note the following identity which follows from (FTI) upon clearing denominators:

$$\begin{aligned}&\frac{\theta(u-u'+x)}{\theta(u-u')\theta(x)}\frac{\theta(v-u'-x+t)}{\theta(v-u')\theta(-x+t)} = \\ &\qquad \frac{\theta(u-v+x)}{\theta(u-v)\theta(x)}\frac{\theta(v-u'+t)}{\theta(v-u')\theta(t)} - \frac{\theta(u-v+x-t)}{\theta(u-v)\theta(x-t)}\frac{\theta(u-u'+t)}{\theta(u-u')\theta(t)}\end{aligned}$$

and letting $t\to 0$ in the equation above, we get

$$\begin{aligned}&\frac{\theta(u-u'+x)}{\theta(u-u')\theta(x)}\frac{\theta(v-u'-x)}{\theta(v-u')\theta(-x)} = \\ &\qquad \frac{\theta(u-v+x)}{\theta(u-v)\theta(x)}\frac{\theta'(v-u')}{\theta(v-u')} - \frac{\theta(u-v+x)}{\theta(u-v)\theta(x)}\frac{\theta'(u-u')}{\theta(u-u')} \\ &\qquad\qquad - \frac{\theta(u-v+x)}{\theta(u-v)\theta(x)}\frac{\theta'(x)}{\theta(x)} + \frac{\theta'(u-v+x)}{\theta(u-v)\theta(x)}\end{aligned}$$

Thus, using the fact that $\lambda_1+\lambda_2=\hbar\mu$, we obtain the following expression for $\overline{\mathcal{T}}$:

- If $\mu_i\neq 0$ then

$$\begin{aligned}\overline{\mathcal{T}} &= \frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\frac{\theta(v-u'+\hbar\mu_i)}{\theta(v-u')\theta(\hbar\mu_i)} + \\ &\qquad \frac{\theta(u-v-\lambda_{2,i})}{\theta(u-v)\theta(\lambda_{2,i})}\frac{\theta(u-u'+\hbar\mu_i)}{\theta(u-u')\theta(\hbar\mu_i)}\end{aligned}$$

- If $\mu_i=0$ then

$$\begin{aligned}\overline{\mathcal{T}} &= \frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\frac{\theta'(v-u')}{\theta(v-u')} - \frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\frac{\theta'(u-u')}{\theta(u-u')} \\ &\qquad - \frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\frac{\theta'(\lambda_{1,i})}{\theta(\lambda_{1,i})} + \frac{\theta'(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\end{aligned}$$

We substitute these expressions for $\overline{\mathcal{T}}$ in

$$\theta(d_i\hbar)[\mathfrak{X}_i^+(u,\lambda_1),\mathfrak{X}_i^-(v,\lambda_2)] = \oint_{\mathcal{C}}\overline{\mathcal{T}}\Phi_i(u')\,du'$$

Note that the last two terms in the formula obtained for $\overline{\mathcal{T}}$ for the case when $\mu_i=0$ will not contribute to the contour integral, since in this case $\Phi_i(u)$ is a doubly–periodic function and hence the sum of its residues will be zero.

Finally the relation ($\mathcal{EQ}5$) follows from the following general result about doubly quasi–periodic functions.

6.9.

Lemma. *Let $f(u)$ be a meromorphic function such that*

$$f(u+1) = f(u) \qquad \textit{and} \qquad f(u+\tau) = e^{-2\pi\iota a} f(u)$$

Choose $S \subset \mathbb{C}$ a (finite) set of representatives modulo Λ_τ of the poles of $f(u)$ and let $\mathcal{C}$ be a contour enclosing S and no other poles of $f(u)$. Then

- *If $a = 0$ then*

$$f(u) = \oint_{\mathcal{C}} \frac{\theta'(u-u')}{\theta(u-u')} f(u')\, du' + K$$

where K is a contant.

- *If $a \notin \Lambda_\tau$ then*

$$f(u) = \oint_{\mathcal{C}} \frac{\theta(u-u'+a)}{\theta(u-u')\theta(a)} f(u')\, du'$$

Proof. This lemma is the analogue of partial fractions for doubly (quasi) periodic functions. The proof is a standard exercise in complex analysis, see *e.g.,* [45, Section 21.50], and is given here solely for completeness.

For each $b \in S$ consider the Laurent series expansion of $f(u)$ near $u = b$:

$$f(u) = \sum_{n\in\mathbb{N}} \frac{f_{b,n}}{(u-b)^{n+1}} + \text{ regular part}$$

Define a new function $\widetilde{f}(u)$ as:

- If $a = 0$ then

$$\widetilde{f}(u) = \sum_{\substack{b\in S\\ n\in\mathbb{N}}} f_{b,n} \frac{(-\partial_u)^n}{n!} \left(\frac{\theta'(u-b)}{\theta(u-b)}\right)$$

- If $a \notin \lambda_\tau$ then

$$\widetilde{f}(u) = \sum_{\substack{b\in S\\ n\in\mathbb{N}}} f_{b,n} \frac{(-\partial_u)^n}{n!} \left(\frac{\theta(u-b+a)}{\theta(u-b)\theta(a)}\right)$$

Note that $\widetilde{f}(u)$ is the right–hand side of the required equations. It remains to observe that both $f(u)$ and $\widetilde{f}(u)$ have the same periodicity properties. Moreover since both $\dfrac{\theta'(x)}{\theta(x)}$ and $\dfrac{\theta(x+a)}{\theta(x)\theta(a)}$ are of the form $x^{-1} + O(1)$ near $x = 0$ (here $O(1)$ stands for an element of $\mathbb{C}[[x]]$), we get that $f(u) - \widetilde{f}(u)$ are holomorphic. The result now follows from the fact that every doubly periodic holomorphic function has to be a constant, and the only double quasi–periodic holomorphic function is zero.

$\square$

7. Factorization problem

7.1. **Set up and statement of the problem.** Let V be a finite–dimensional vector space over $\mathbb{C}$ and $K \in \mathrm{Aut}(V)$ be an invertible operator. Assume given a meromorphic function $\Phi : \mathbb{C}^\times \to \mathrm{End}(V)$ such that

$$[K, \Phi(z)] = 0 \qquad \text{and} \qquad \Phi(pz) = K^{-2}\Phi(z)$$

where recall that $p = e^{2\pi\iota\tau}$ is a non–zero complex number with $|p| < 1$.

Factorization problem. Find $H^{\pm}(z)$ meromorphic $\mathrm{End}(V)$–valued functions of a complex variable $z \in \mathbb{C}^\times$ such that

(F1) $\Phi(z) = H^+(z)^{-1}H^-(z)$.
(F2) $[K, H^{\pm}(z)] = 0$.
(F3) $H^{\pm}(z)$ are holomorphic and invertible in a neighborhood of $z = 0, \infty$ respectively.
(F4) $H^-(\infty) = 1$.

Note that we do not impose any normalization condition on $H^+(z)$.

7.2. **Coefficient matrix.** Assuming $H^{\pm}$ is a solution to the factorization problem stated in section 7.1, we define $\overline{A}(z) := H^-(pz)H^-(z)^{-1}$. By the periodicity condition on $\Phi(z)$ and (F1), we have

$$\overline{A}(z) = K^{-2}H^+(pz)H^+(z)^{-1} = H^-(pz)H^-(z)^{-1}$$

Since $H^{\pm}$ are regular near $z = 0, \infty$ respectively, $\overline{A}(z)$ is a meromorphic function on $\mathbb{P}^1$ and hence a rational function of z. Moreover, we have

$$\overline{A}(0) = K^{-2} \qquad \text{and} \qquad \overline{A}(\infty) = 1$$

Let $A(z) = K\overline{A}(z)$ so that $A(\infty) = K = A(0)^{-1}$. We refer to $A(z)$ as the coefficient matrix.

7.3. **Isomonodromy transformation.** If $H_1^{\pm}$ and $H_2^{\pm}$ are two solutions of the factorization problem (section 7.1), then we have

$$H_2^+(z)H_1^+(z)^{-1} = H_2^-(z)H_1^-(z)^{-1}$$

Since the left–hand side of this equation is regular at 0 and the right–hand side is regular at ∞, the resulting function is again a rational function. Let us denote it by $G(z) = H_2^{\pm}(z)H_1^{\pm}(z)^{-1}$. This rational function is regular at both 0 and ∞ and $G(\infty) = 1$.

Let $A_1(z)$ and $A_2(z)$ be the cofficient matrices defined using the solutions $H_1^{\pm}$ and $H_2^{\pm}$ of the factorization problem, respectively. Then we get

$$\begin{aligned} A_2(z) &= KH_2^-(pz)H_2^-(z)^{-1} = G(pz)KH_1^-(pz)H_1^-(z)^{-1}G(z)^{-1} \\ &= G(pz)A_1(z)G(z)^{-1} \end{aligned}$$

Given two subsets $S_1, S_2 \in \mathbb{C}^\times$, we call the pair (S_1, S_2) *non–congruent* if $\alpha_1\alpha_2^{-1} \notin p^{\mathbb{Z}_{\neq 0}}$ for any $\alpha_s \in S_s$, $s = 1, 2$. We shall also say that $S \subset \mathbb{C}^\times$ is non–congruent if it is non–congruent to itself. We have the following analog of [30, Proposition 4.11].

Proposition. *Let* $\mathcal{P}_s, \mathcal{Z}_s \subset \mathbb{C}^\times$ *be the set of poles* $A_s(z)$, $A_s(z)^{-1}$ *respectively* ($s = 1, 2$)*. If the pairs*

$$(\mathcal{Z}_1, \mathcal{P}_1),\ (\mathcal{Z}_2, \mathcal{P}_2),\ (\mathcal{Z}_1, \mathcal{Z}_2),\ (\mathcal{P}_1, \mathcal{P}_2)$$

are non–congruent, then $A_1 = A_2$*. And therefore,* $H_1^\pm = H_2^\pm$*.*

7.4. **Existence of a factorization in the abelian case.** Now we turn our attention to finding a solution to the factorization problem, under the assumption that $[\Phi(z), \Phi(w)] = 0$, for every $z, w \in \mathbb{C}^\times$. Using Lemmas 4.12 and 4.13 of [30], we have the Jordan decomposition of $\Phi(z) = \Phi_S(z)\Phi_U(z)$. Let $\mathcal{P}, \mathcal{P}_S, \mathcal{P}_U$ be the sets of poles of $\Phi(z), \Phi_S(z), \Phi_U(z)$ respectively, and $\mathcal{Z}, \mathcal{Z}_S, \mathcal{Z}_U$ those of $\Phi(z)^{-1}, \Phi_S(z)^{-1}, \Phi_U(z)^{-1}$ respectively. Then we have the following (see [30, Lemma 4.13])

(1) $\mathcal{P} = \mathcal{P}_S \cup \mathcal{P}_U$ and $\mathcal{Z} = \mathcal{Z}_S \cup \mathcal{Z}_U$.
(2) $\mathcal{P}_U = \mathcal{Z}_U$.

We provide a solution to the factorization problem in the semisimple and unipotent cases respectively in Sections 7.5 and 7.6 below.

7.5. **Semisimple case.** Assuming both K and $\Phi(z)$ are semisimple, we can restrict to their joint eigenspace which reduces the problem to the scalar case. Let $\eta \in \mathbb{C}^\times$ and $\varphi(z)$ a meromorphic function be a joint eigenvalue of $K, \Phi(z)$. Then by general theory of elliptic functions we have

$$\varphi(z) = C \prod_l \left.\frac{\theta(u - a_l)}{\theta(u - b_l)}\right|_{z = e^{2\pi\iota u}}$$

where by the quasi–periodicity of $\Phi(z)$ we have

$$\eta = \exp\left(\pi\iota \sum_l (b_l - a_l)\right) \tag{7.1}$$

Now using the expression (2.3) for the theta function, we have the following solution to the factorization problem:

$$H^+(z)^{-1} = C \prod_l \left(\frac{z - \alpha_l}{z - \beta_l}\right) \frac{\theta^+(u - a_l)}{\theta^+(u - b_l)}$$
$$H^-(z) = \prod_l \frac{\theta^-(u - a_l)}{\theta^-(u - b_l)}$$

where the change of variable $z = e^{2\pi\iota u}$ is understood.

7.6. **Unipotent case.** Now we assume that both $\Phi(z)$ and K are unipotent. This enables us to take logarithm in order to convert the problem to an additive one. Namely, we have the following periodicity property

$$\log(\Phi(pz)) - \log(\Phi(z)) = \log(K^{-2}) \tag{7.2}$$

and we are required to find $h^\pm(z)$, regular at 0 and ∞ respectively, such that $h^-(\infty) = 0$ and

$$\log(\Phi(z)) = h^-(z) - h^+(z)$$

The problem again reduces to the one for a single function. Let $\varphi(z)$ be an entry of the matrix $\log(\Phi(z))$ and $k \in \mathbb{C}$ be the correspoding entry of $\log(K^{-2})$, so that we have $\varphi(pz)-\varphi(z)=k$. Again, from the general theory of elliptic functions, we have the following expression for $\varphi(z)$:

$$\varphi(z)=\sum_{\substack{a\in\mathcal{P}\\ n\in\mathbb{N}}} f_{a,n}\frac{\partial_u^{n+1}}{(n+1)!}\log(\theta(u-a))$$

where $\mathcal{P}\subset\mathbb{C}$ is a choice modulo Λ_τ of representatives of poles of $\varphi\left(e^{2\pi\iota u}\right)$, and $-2\pi\iota\sum_a f_{a,0}=k$.

Using the equation (2.3) we obtain the following solution

$$h^+(z)=-\sum_{\substack{a\in\mathcal{P}\\ n\in\mathbb{N}}} f_{a,n}\frac{\partial_u^{n+1}}{(n+1)!}\log((e^{\pi\iota u}-e^{-\pi\iota u})\theta^+(u-a))$$

$$h^-(z)=\sum_{\substack{a\in\mathcal{P}\\ n\in\mathbb{N}}} f_{a,n}\frac{\partial_u^{n+1}}{(n+1)!}\log(\theta^-(u-a))$$

Note that $h^+(0)=-\pi\iota\sum_{a\in\mathcal{P}} f_{a,0}=k/2$.

7.7. Let us define $G^-(z)=H^-(z)$ and $G^+(z)=H^+(z)^{-1}A(z)^{-1}$, where $A(z)$ was defined in Section 7.2, so that $\Phi(z)=G^+(z)A(z)G^-(z)$. As a consequence of the explicit factorization given in the previous two sections we have the following:

Corollary. *$G^+(0)$ is semisimple. More precisely, if $V'\subset V$ is a generalized eigenspace for $\Phi(z)$ with the following generalized eigenvalue*

$$\varphi(z)=C\prod_l\frac{\theta(u-a_l)}{\theta(u-b_l)}$$

Then for the solution given in Section 7.5, $G^+(0)=C$ on V'.

8. The inverse functor

8.1. From now on we fix a subset $\mathrm{S}\subset\mathbb{C}$ subject to the following conditions

- $a\neq b\in\mathrm{S}\Rightarrow a-b\notin\Lambda_\tau$.
- $\mathrm{S}\pm\frac{\hbar}{2}=\mathrm{S}$.
- S is in bijection with $\mathbb{C}/\Lambda_\tau$ under the natural surjection $\mathbb{C}\to\mathbb{C}/\Lambda_\tau$.

Let us assume that $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ is a λ–flat object. We consider the factorization problem for the functions $\Phi_i(u)\in\mathrm{End}(\mathbb{V})$. Using Proposition 7.3, Section 7.4 and Proposition 4.11 we get

Proposition. *There exist unique functions $G_i^\pm(z),\Psi_i(z)\in\mathrm{End}(\mathbb{V})$ such that*

1. *$\Phi_i(u)=G_i^+(z)\Psi_i(z)G_i^-(z)$ where $z=e^{2\pi\iota u}$.*
2. *$G_i^\pm(z)$ are holomorphic and invertible near $z=0,\infty$ respectively. $\Psi_i(z)$ is a rational function of z regular at both $z=0$ and ∞.*
3. *$G_i^-(\infty)=1$.*

(4)
$$\sigma(G_i^{\pm}(z)) \subset \bigcup_{n\geq 1} p^{\mp n}\, \mathrm{S}^* \quad \textit{and } \sigma(\Psi_i(z)) \subset \mathrm{S}^*$$

where $\mathrm{S}^* = \{\exp(2\pi\iota s)\}_{s\in \mathrm{S}}$.

8.2. Combining the results Proposition 8.1, 4.11 and Corollary 7.7 we get that for every generalized eigenspace $\mathbb{V}[\underline{\mathcal{B}}]$ the functions $B_i(u)$ have the following form:

$$B_i(u) = C_i(\underline{\mathcal{B}}) \prod_k \frac{\theta(u - c_{i,k} + d_i\hbar)}{\theta(u - c_{i,k})} \prod_l \frac{\theta(u - c'_{i,l} - d_i\hbar)}{\theta(u - c'_{i,l})} \tag{8.1}$$

where the numbers $c_{i,k}, c'_{i,l} \in \mathrm{S}$.

And the matrix $G_i^+(0)$ is given by:

$$G_i^+(0) = \sum_{\underline{\mathcal{B}}} C_i(\underline{\mathcal{B}})\, \mathrm{Id}_{\mathbb{V}[\underline{\mathcal{B}}]} \tag{8.2}$$

8.3. **Second gauge transformation.** Consider the following zero–weight End($\mathbb{V}$)–valued function, which is given by the following expression on a weight space $\mathbb{V}_\mu$

$$\varphi(\lambda) = \exp\left(\frac{1}{\hbar}\sum_{j\in\mathbf{I}}\left(\lambda + \frac{\hbar}{2}\mu, \varpi_j^\vee\right)\ln(G_j^+(0))\right)$$

We conjugate $\mathbb{V}$ by the automorphism given above to obtain $\mathbb{V}^\varphi$ which is manifesly isomorphic to $\mathbb{V}$ with isomorphism given by $\varphi(\lambda)$. Again $\mathbb{V}^\varphi = \mathbb{V}$ as $\mathfrak{h}$–diagonalizable module and the action of $\{\Phi_i(u), \mathfrak{X}_i^{\pm}(u,\lambda)\}$ on a weight space $\mathbb{V}_\mu^\varphi$ is given by:

$$\Phi_i^\varphi(u) = \varphi\left(\lambda + \frac{\hbar}{2}\alpha_i\right)^{-1} \Phi_i(u)\varphi\left(\lambda - \frac{\hbar}{2}\alpha_i\right)$$
$$\mathfrak{X}_i^{\pm,\varphi}(u,\lambda) = \varphi\left(\pm\lambda \mp \frac{\hbar}{2}(\mu \pm \alpha_i) + \frac{\hbar}{2}\alpha_i\right)^{-1} \mathfrak{X}_i^{\pm}(u,\lambda)$$
$$.\varphi\left(\pm\lambda \mp \frac{\hbar}{2}\mu - \frac{\hbar}{2}\alpha_i\right)$$

Proposition. $\mathbb{V}^\varphi \in \mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ *and we have the following*

(1) *The generalized eigenvalues of* $\Phi_i^\varphi(u)$ *have the following form*
$$\prod_k \frac{\theta(u - c_{i,k} + d_i\hbar)}{\theta(u - c_{i,k})} \prod_l \frac{\theta(u - c'_{i,l} - d_i\hbar)}{\theta(u - c'_{i,l})}$$
where the numbers $c_{i,k}, c'_{i,l} \in \mathrm{S}$.

(2) $G_i^{+,\varphi}(0) = 1 = G^{-,\varphi}(\infty)$.

(3) *For each* $i \in \mathbf{I}$, $b \in \mathrm{S}$ *and* $n \in \mathbb{N}$, $\mathsf{X}_{i;b,n}^{\pm,\varphi}(\lambda)$ *is independent of* λ.

Proof. We begin by computing the action of $\Phi_i^{\varphi}(u)$. Consider a generalized eigenspace $\mathbb{V}[\underline{\mathcal{B}}]$ where $\underline{\mathcal{B}}$ consists of $\mathbf{I}$–tuple of functions $(B_i(u))$ given by (8.1). On this generalized eigenspace $\Phi_i^{\varphi}(u)$ acts by:

$$\begin{aligned}\Phi_i^{\varphi}(u)|_{\mathbb{V}[\underline{\mathcal{B}}]} &= \exp\left(\frac{1}{\hbar}\sum_{j\in\mathbf{I}}(-\hbar\alpha_i,\varpi_j^{\vee})\ln(C_i(\underline{\mathcal{B}}))\right)\Phi_i(u)|_{\mathbb{V}[\underline{\mathcal{B}}]}\\ &= C_i(\underline{\mathcal{B}})^{-1}\ \Phi_i(u)|_{\mathbb{V}[\underline{\mathcal{B}}]}\end{aligned}$$

This implies that the shifted conjugation by φ of $\Phi_i(u)$ is independent of λ, as we require. Moreover we obtain (1) and (2) of the Proposition.

Let us consider the raising and lowering operators now. Using Proposition 3.5 we know that between two generalized eigenspaces $\mathbb{V}_{\mu}[\underline{\mathcal{B}}]$ and $\mathbb{V}_{\mu\pm\alpha_j}[\underline{\mathcal{B}}^{\pm}]$ these operators are of the form:

$$\mathsf{X}^{\pm}_{j;b,n}(\lambda) = e^{\pm 2\pi\iota(\lambda,\alpha^{\pm})}\mathsf{X}^{\pm}_{j;b,n}(0)$$

and in this case the constants from (8.1) are related by

$$C_i(\underline{\mathcal{B}}^{\pm}) = C_i(\underline{\mathcal{B}})e^{2\pi\iota\hbar(\alpha_i,\alpha^{\pm})}$$

With this information in mind, we can compute the shifted conjugation.

$$\begin{aligned}\mathsf{X}^{\pm,\varphi}_{j;b,n}(\lambda) &= C^{\pm}\exp\left(\frac{1}{\hbar}\sum_{k\in\mathbf{I}}2\pi\iota\hbar(\alpha^{\pm},\alpha_k)(\mp\lambda,\varpi_k^{\vee})\right)e^{\pm 2\pi\iota(\lambda,\alpha^{\pm})}\mathsf{X}^{\pm}_{j;b,n}(0)\\ &= C^{\pm}e^{\mp 2\pi\iota(\lambda,\alpha^{\pm})}e^{\pm 2\pi\iota(\lambda,\alpha^{\pm})}\mathsf{X}^{\pm}_{j;b,n}(0)\\ &= C^{\pm}\mathsf{X}^{\pm}_{j;b,n}(0)\end{aligned}$$

where we have used the fact that $\lambda \in \overline{\mathfrak{h}^*} = \sum_{i\in\mathbf{I}}\mathbb{C}\alpha_i$ to conclude that

$$\sum_{k\in\mathbf{I}}(\lambda,\varpi_k^{\vee})(\alpha,\alpha_k) = (\lambda,\alpha)$$

for every $\alpha \in Q$. The constants $C^{\pm}$ appearing above are independent of λ. They can be easily computed from the definition:

$$\begin{aligned}C^{+} &= C_j(\underline{\mathcal{B}})^{-1}\exp\left(-\pi\iota\hbar(\alpha_j,\alpha^{+})\right)\\ C^{-} &= C_j(\underline{\mathcal{B}})^{-2}\exp\left(-3\pi\iota\hbar(\alpha_j,\alpha^{-})\right)\exp\left(-2\pi\iota\hbar\sum_{k\in\mathbf{I}}(\mu,\varpi_k^{\vee})(\alpha^{-},\alpha_k)\right)\end{aligned}$$

This calculation together with the partial fraction expression of the raising and lowering operators from Lemma 3.2 implies both (3) and the fact that the resulting representation $\mathbb{V}^{\varphi}$ is in $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. □

8.4. **Definition of category $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$.** Let $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ be the full subcategory of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ consisting of $\mathbb{V}$ which satisfies (1) of Proposition 8.3. That is, if for each $i \in \mathbf{I}$, the generalized eigenvalues of $\Phi_i(u)$ acting on $\mathbb{V}$ are of the form:

$$\prod_k \frac{\theta(u - c_{i,k} + d_i\hbar)}{\theta(u - c_{i,k})}\prod_l \frac{\theta(u - c'_{i,l} + d_i\hbar)}{\theta(u - c'_{i,l})}$$

where the numbers $c_{i,k}, c'_{i,l}$ are in S. For future referece, we record the following consequence of Proposition 8.3

Corollary. *Every object of* $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ *is isomorphic to some object of* $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$.

Later we will also need a $\mathbb{C}$–linear analogue of $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ which we define as follows. Let $\mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ be the category whose objects are same as those of $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ while its morphisms are the usual $\mathbb{C}$–linear homomorphisms commuting with the action of $\mathfrak{h}$ and $\{\Phi_i(u), \mathfrak{X}^{\pm}_i(u,\lambda)\}$.

Following [30, Section 3.5] we consider the full category $\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ of $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ consisting of $\mathcal{V}$ for which the poles of $\{\Psi_i(z)^{\pm 1}\}$ lie in $\exp(2\pi\iota\,\mathrm{S})$. We refer the reader to [30, Theorem 3.8] for several equivalent characterizations of $\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$.

8.5. **Choice of contours.** Since $\mathbb{V}$ and $\mathbb{V}^{\varphi}$ are isomorphic, we identify the two and thus omit the superscript φ. In the next section we define an action of $U_q(L\mathfrak{g})$ on $\mathbb{V}$ which relies upon a choice of contours. Let $\mu \in \mathfrak{h}^*$ be a weight of $\mathbb{V}$ and $i \in \mathbf{I}$. Let $\mathcal{C}^{\pm}_{i,\mu}$ be a Jordan curve such that

- $\mathcal{C}^{\pm}_{i,\mu}$ encloses all the poles of $\mathfrak{X}^{\pm}_i(u,\lambda)$ in the spectral variable u, which are contained in S, and none of their translates under $\Lambda_\tau \setminus \{0\}$.
- $\mathcal{C}^{\pm}_{i,\mu}$ does not enclose any $\Lambda_\tau \setminus \{0\}$ translates of the poles of $\Phi_i(u)^{-1}_{\mu}$.

Note that such a curve exists since the poles of $\Phi_i(u)^{-1}$ are contained in S by Proposition 4.11, and the latter is non–congruent.

8.6. **Inverse construction.** Let $\Psi_i(z), G^{\pm}_i(z)$ be the $\mathrm{End}(\mathbb{V})$–valued functions given by Proposition 8.1. Let us write $\mathcal{V} = \Xi(\mathbb{V}) = \mathbb{V}$ as $\mathfrak{h}$–diagonalizable module and define operators $\Psi_i(z), \mathcal{X}^{\pm}_i(z)$ on the weight space $\mathcal{V}_\mu$ as:

(1) $\Psi_i(z)$ as given by Proposition 8.1.

(2) For each $i \in \mathbf{I}$

$$\mathcal{X}^{\pm}_i(z) := \frac{1}{c^{\pm}_i} \int_{\mathcal{C}^{\pm}_{\mu}} \frac{z}{z - e^{2\pi\iota u}} G^{\pm}_i \left(e^{2\pi\iota u}\right)^{-1} \mathfrak{X}^{\pm}_i(u,\lambda)\, du \tag{8.3}$$

where the contours $\mathcal{C}^{\pm}_{i,\mu}$ are chosen as in section 8.5 and the constants $c^{\pm}_i$ are as in (6.4). Note that by (3) of Proposition 8.3 the resulting operators are independent of λ.

Theorem. *The operators* $\Psi_i(z), \mathcal{X}^{\pm}_i(z)$ *defined above satisfy the relations of* $U_q(L\mathfrak{g})$ *given in Section 5.2 and hence define a representation* $\Xi(\mathbb{V}) \in \mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$. *Moreover* $\Theta(\Xi(\mathbb{V})) = \mathbb{V}$ *and for any* $\mathcal{V} \in \mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ *we have* $\Xi(\Theta(\mathcal{V})) = \mathcal{V}$. *In particular, the categories* $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ *and* $\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ *are isomorphic.*

Proof. We need to verify the axioms of Section 5.2. Note that the category $\mathcal{O}$ and integrability condition holds by the corresponding assumption on $\mathbb{V}$. The normalization condition follows from Section 7.2 and the definition of $\mathcal{X}^{\pm}_i(z)$.

The relations ($\mathcal{QL}1$) and ($\mathcal{QL}2$) are immediate. After working out the commutation relations between $G^{\pm}_i(z)$ and operators defined using contour integration involvoing $\mathfrak{X}_i^{\pm}(u,\lambda)$ in Section 8.7, we prove the relations ($\mathcal{QL}3$), ($\mathcal{QL}4$) and ($\mathcal{QL}5$) in Sections 8.8, 8.9 and 8.10 respectively.

The fact that the commuting operators on $\mathbb{V}$ and $\mathcal{V}$ are the same follows from the uniqueness statement of Proposition 7.3. The proof for the raising and lowering operators is given in Sections 8.11 and 8.12. $\square$

Restricting to the category $\mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ defined in Section 8.4, we obtain the following.

Corollary. *The functors* $\Theta^{\mathrm{S}}:\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))\to\mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$ *and* $\Xi^{\mathrm{S}}:\mathsf{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})\to\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ *are inverse to each other and hence the two categories are isomorphic.*

8.7. **Commutation relations.** Now let $i,j\in\mathbf{I}$ and $a=d_ia_{ij}\hbar/2$. Consider a contour $\mathcal{C}$ with interior domain D and $\Omega_1,\Omega_2\subset\mathbb{C}$ two open subsets with $\overline{D}\subset\Omega_2$. Assume given a holomorphic function $f(u,v):\Omega_1\times\Omega_2\to\mathrm{End}(\mathbb{V})$ such that $[\Phi_i(u),f(u,v)]=0$ for any u,v.

Proposition. *For each $\epsilon\in\{\pm1\}$ we have:*

(1) *If $u\notin\overline{D}\pm\epsilon a+\mathbb{Z}$, then*

$$\mathrm{Ad}(\Psi_i\left(e^{2\pi\iota u}\right))^{\pm1}\oint_{\mathcal{C}}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv=\oint_{\mathcal{C}}\left(\frac{e^{2\pi\iota(u+\epsilon a)}-e^{2\pi\iota v}}{e^{2\pi\iota u}-e^{2\pi\iota(v+\epsilon a)}}\right)^{\pm1}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv$$

(2) *For $u\notin\overline{D}+\mathbb{Z}\pm\epsilon a+\mathbb{Z}_{<0}\tau$, we have*

$$\mathrm{Ad}(G_i^{+}\left(e^{2\pi\iota u}\right))^{\pm1}\oint_{\mathcal{C}}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv=\oint_{\mathcal{C}}\left(\frac{\theta^{+}(u-v+\epsilon a)}{\theta^{+}(u-v-\epsilon a)}\right)^{\pm1}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv$$

(3) *For $u\notin\overline{D}+\mathbb{Z}\pm\epsilon a+\mathbb{Z}_{>0}\tau$, we have*

$$\mathrm{Ad}(G_i^{-}\left(e^{2\pi\iota u}\right))^{\pm1}\oint_{\mathcal{C}}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv=\oint_{\mathcal{C}}\left(\frac{\theta^{-}(u-v+\epsilon a)}{\theta^{-}(u-v-\epsilon a)}\right)^{\pm1}f(u,v)\mathfrak{X}_j^{\epsilon}(v,\lambda)\,dv$$

8.8. **Proof of ($\mathcal{QL}3$).** Let $i,j\in\mathbf{I}$ and set $a=\hbar d_ia_{ij}/2$. We need to prove that

$$\mathrm{Ad}(\Psi_i(z))\mathcal{X}_j^{\pm}(w)\;=\;\frac{q_i^{\pm a_{ij}}z-w}{z-q_i^{\pm a_{ij}}w}\mathcal{X}_j^{\pm}(w)\;-\;\frac{(q_i^{\pm a_{ij}}-q^{\mp a_{ij}})q^{\pm a_{ij}}w}{z-q^{\pm a_{ij}}w}\mathcal{X}_j^{\pm}(q_i^{\mp a_{ij}}z)$$

Using the definition (8.3) and (1) of Proposition 8.7 the left–hand side of this equation becomes

$$\text{L.H.S.}\;=\frac{1}{c_j^{\pm}}\int_{\mathcal{C}_j}\frac{q_i^{\pm a_{ij}}z-w'}{z-q_i^{\pm a_{ij}}w'}\frac{w}{w-w'}G_j^{\pm}\left(e^{2\pi\iota u}\right)^{-1}\mathfrak{X}_j^{\pm}(u,\lambda)\,du$$

where for notational convenience we wrote $w'=e^{2\pi\iota u}$.

Similarly the right–hand side takes the following form

$$\text{R.H.S.}\;=\frac{1}{c_j^{\pm}}\int_{\mathcal{C}_j}\mathcal{K}(z,w,u)G_j^{\pm}\left(e^{2\pi\iota u}\right)^{-1}\mathfrak{X}_j^{\pm}(u,\lambda)\,du$$

where

$$\mathcal{K}(z,w,u)=\frac{q_i^{\pm a_{ij}}z-w}{z-q_i^{\pm a_{ij}}w}\frac{w}{w-w'}-\frac{(q_i^{\pm a_{ij}}-q^{\mp a_{ij}})q^{\pm a_{ij}}w}{z-q^{\pm a_{ij}}w}\frac{z}{z-q_i^{\pm a_{ij}}w'}$$

Thus the relation ($\mathcal{QL}3$) reduces to the following algebraic identity which can be checked easily:

$$\mathcal{K}(z,w,u) = \frac{q_i^{\pm a_{ij}} z - w'}{z - q_i^{\pm a_{ij}} w'} \frac{w}{w-w'}$$

8.9. **Proof of ($\mathcal{QL}4$).** For $i,j \in \mathbf{I}$, let $a = \hbar d_i a_{ij}/2$. We need to prove the following relation:

$$\begin{aligned}&(z - q_i^{\pm a_{ij}} w)\mathcal{X}_i^{\pm}(z)\mathcal{X}_j^{\pm}(w) - z\mathcal{X}_i^{\pm}(\infty)\mathcal{X}_j^{\pm}(w) + q_i^{\pm a_{ij}} w\mathcal{X}_i^{\pm}(z)\mathcal{X}_j^{\pm}(\infty) = \\ &\quad (q_i^{\pm a_{ij}} z - w)\mathcal{X}_j^{\pm}(w)\mathcal{X}_i^{\pm}(z) - q_i^{\pm a_{ij}} z\mathcal{X}_j^{\pm}(w)\mathcal{X}_i^{\pm}(\infty) + w\mathcal{X}_j^{\pm}(\infty)\mathcal{X}_i^{\pm}(z)\end{aligned}$$

Let us take $\lambda \in \mathfrak{h}^*$ and let

$$\lambda_1 = \lambda \pm \frac{\hbar}{2}\alpha_j \text{ and } \lambda_2 = \lambda \mp \frac{\hbar}{2}\alpha_i$$

Again using (2) and (3) of Proposition 8.7 and the definition (8.3) we find that the left–hand side multiplied by $c_i^{\pm} c_j^{\pm}$ is given by

$$\iint \mathcal{I}_1(z,w,u,v) G_i^{\pm}\left(e^{2\pi\iota u}\right)^{-1} G_j^{\pm}\left(e^{2\pi\iota v}\right)^{-1} \mathfrak{X}_i^{\pm}(u,\lambda_1)\mathfrak{X}_j^{\pm}(v,\lambda_2)\, du\, dv$$

where (we write $z' = e^{2\pi\iota u}$ and $w' = e^{2\pi\iota v}$):

$$\mathcal{I}_1 = \frac{zw(z' - q_i^{\pm a_{ij}} w'}{(z-z')(w-w')} \frac{\theta^{\pm}(v-u\pm a)}{\theta^{\pm}(v-u\mp a)}$$

Similarly the right–hand side multiplied by $c_i^{\pm} c_j^{\pm}$ equals

$$\iint \mathcal{I}_2(z,w,u,v) G_i^{\pm}\left(e^{2\pi\iota u}\right)^{-1} G_j^{\pm}\left(e^{2\pi\iota v}\right)^{-1} \mathfrak{X}_j^{\pm}(v,\lambda_2)\mathfrak{X}_i^{\pm}(u,\lambda_1)\, du\, dv$$

where

$$\mathcal{I}_2 = \frac{zw(q_i^{\pm a_{ij}} z' - w'}{(z-z')(w-w')} \frac{\theta^{\pm}(u-v\pm a)}{\theta^{\pm}(u-v\mp a)}$$

Using relation ($\mathcal{EQ}4$) we can flip the $\mathfrak{X}_j$ and $\mathfrak{X}_i$ factors to get that the right–hand side (multiplied by $c_i^{\pm} c_j^{\pm}$) is equal to

$$\iint \mathcal{I}_2(z,w,u,v) \frac{\theta(u-v\mp a)}{\theta(u-v\pm a)} G_i^{\pm}\left(e^{2\pi\iota u}\right)^{-1} G_j^{\pm}\left(e^{2\pi\iota v}\right)^{-1} \mathfrak{X}_i^{\pm}(u,\lambda_1)\mathfrak{X}_j^{\pm}(v,\lambda_2)\, du\, dv$$

Hence, in order to prove the relation ($\mathcal{QL}4$), we need to check the following identity

$$\frac{\theta(u-v+b)}{\theta(u-v-b)} = \frac{e^{2\pi\iota u} - e^{2\pi\iota(v+b)}}{e^{2\pi\iota(u+b)} - e^{2\pi\iota v}} \frac{\theta^{+}(u-v+b)}{\theta^{+}(u-v-b)} \frac{\theta^{-}(u-v+b)}{\theta^{-}(u-v-b)}$$

which follows directly from (2.3).

8.10. **Proof of ($\mathcal{QL}$5).** We will prove the following relation, for each $i,j\in\mathbf{I}$ and $k,l\in\mathbb{Z}$:

$$[\mathcal{X}_{i,k}^+,\mathcal{X}_{j,l}^-]=\delta_{ij}\frac{\Psi_{i,k+l}^+-\Psi_{i,k+l}^-}{q_i-q_i^{-1}}$$

on a weight space $\mathcal{V}_\mu$. Let $\lambda\in\mathfrak{h}^*$ and let us take $\lambda_1=\lambda$ and $\lambda_2=-\lambda+\hbar\mu$. Again from Proposition 8.7 and definition 8.3 we get that the left–hand side is equal to

$$\text{L.H.S}=\frac{1}{c_i^+c_j^-}\iint e^{2\pi\iota(ku+lv)}\frac{\theta^+(u-v-a)}{\theta^+(u-v+a)}G_i^+\left(e^{2\pi\iota u}\right)^{-1}G_j^-\left(e^{2\pi\iota v}\right)^{-1}\cdot \left[\mathfrak{X}_i^+(u,\lambda_1),\mathfrak{X}_j^-(v,\lambda_2)\right]\,du\,dv$$

Therefore, it is zero for $i\neq j$ by relation ($\mathcal{EQ}$5). We now assume that $i=j$ and continue with the proof. Let $c=1/(c_i^+c_i^-\theta(d_i\hbar))$. Then

$$\text{L.H.S}=c\int_{\mathcal{C}}\int_{\mathcal{C}}e^{2\pi\iota(ku+lv)}\frac{\theta^+(u-v-d_i\hbar)}{\theta^+(u-v+d_i\hbar)}G_i^+\left(e^{2\pi\iota u}\right)^{-1}G_i^-\left(e^{2\pi\iota v}\right)^{-1}\cdot \left(\frac{\theta(u-v+\lambda_{1,i})}{\theta(u-v)\theta(\lambda_{1,i})}\Phi_i(v)+\frac{\theta(u-v-\lambda_{2,i})}{\theta(u-v)\theta(\lambda_{2,i})}\Phi_i(u)\right)du\,dv$$

To continue with the calculation we use the same technique as in [30, Lemma 5.8]. Namely let $\mathcal{C}_<$ be a small deformation of $\mathcal{C}$ contained in $\mathcal{C}$. Then we integrate with respect to u first over the contour $\mathcal{C}_<$, to get:

$$\text{L.H.S}=c\int_{\mathcal{C}}\int_{\mathcal{C}_<}e^{2\pi\iota(ku+lv)}\frac{\theta^+(u-v-d_i\hbar)}{\theta^+(u-v+d_i\hbar)}G_i^+\left(e^{2\pi\iota u}\right)^{-1}G_i^-\left(e^{2\pi\iota v}\right)^{-1}\cdot \left(\frac{\theta(u-v-\lambda_{2,i})}{\theta(u-v)\theta(\lambda_{2,i})}\Phi_i(u)\right)du\,dv$$

Now we integrate this term with respect to the variable v. Note that the integrand only has simple poles at $v=u$ within $\mathcal{C}$. Using the fact that θ is an odd function, we get

$$\begin{aligned}\text{L.H.S}&=c(2\pi\iota)\int_{\mathcal{C}_<}e^{2\pi\iota(k+l)u}\frac{\theta^+(-d_i\hbar)}{\theta^+(+d_i\hbar)}G_i^+\left(e^{2\pi\iota u}\right)^{-1}G_i^-\left(e^{2\pi\iota u}\right)^{-1}\Phi_i(u)\,du\\&=c'\int_{\mathcal{C}_<}e^{2\pi\iota(k+l)u}\Psi_i\left(e^{2\pi\iota u}\right)\,du\\&=c'\oint_{\mathcal{C}_<}z^{k+l-1}\Psi_i(z)\,dz\\&=\frac{\Psi_{i,k+l}^+-\Psi_{i,k+l}^-}{q_i-q_i^{-1}}\end{aligned}$$

where we have used the fact that $\Phi_i=G_i^+\Psi_iG_i^-$ in the second line, the substitution $z=e^{2\pi\iota u}$ in the third line and we have written $c'=2\pi\iota c\theta^-(d_i\hbar)/\theta^+(d_i\hbar)$. Note that by (6.4) we get

$$c'=\frac{2\pi\iota}{\theta(d_i\hbar)}\frac{\theta^+(d_i\hbar)^2}{(2\pi\iota)^2\theta^+(0)^2}\frac{\theta^-(d_i\hbar)}{\theta^+(d_i\hbar)}=\frac{1}{q_i-q_i^{-1}}$$

by (2.3).

8.11. **Proof of $\Theta\circ\Xi = \mathrm{Id}$.** Let $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. By definition (6.3), $\mathfrak{X}_i^{\pm}(u,\lambda)$ acts on $\Theta\circ\Xi(\mathbb{V})$ by the following

$$
\begin{aligned}
&c_i^{\pm}\oint_{\mathcal{C}}\frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)}G_i^{\pm}\left(e^{2\pi\iota v}\right)\mathcal{X}_i^{\pm}\left(e^{2\pi\iota v}\right)\,dv\\
&=\oint_{\mathcal{C}}\int_{\mathcal{C}'}\frac{\theta(u-v+\lambda_i)}{\theta(u-v)\theta(\lambda_i)}G_i^{\pm}\left(e^{2\pi\iota v}\right)\frac{e^{2\pi\iota v}}{e^{2\pi\iota v}-e^{2\pi\iota v'}}G_i^{\pm}\left(e^{2\pi\iota v'}\right)^{-1}\mathfrak{X}_i^{\pm}(v',\lambda)\,dv\,dv'
\end{aligned}
$$

Assuming $\mathcal{C}'$ is contained in $\mathcal{C}$, we integrate with respect to v first to get:

$$
\frac{1}{2\pi\iota}\int_{\mathcal{C}'}\frac{\theta(u-v'+\lambda_i)}{\theta(u-v')\theta(\lambda_i)}\mathfrak{X}_i^{\pm}(v',\lambda)\,dv'
$$

This is same as $\mathfrak{X}_i^{\pm}(u,\lambda)$, since both functions have same quasi–periodicity with respect to shifts under τ and their difference has no poles by our choice of the contour $\mathcal{C}'$ in the definition (8.3). Therefore this difference must be zero.

8.12. **Proof of $\Xi\circ\Theta = \mathrm{Id}$.** Let $\mathcal{V}\in\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$. By definition (8.3) the operators $\mathcal{X}_i^{\pm}(z)$ act on $\Xi\circ\Theta(\mathcal{V})$ by the following

$$
\begin{aligned}
&\frac{1}{c_i^{\pm}}\int_{\mathcal{C}}\frac{z}{z-e^{2\pi\iota v}}G_i^{\pm}\left(e^{2\pi\iota v}\right)^{-1}\mathfrak{X}_i^{\pm}(v,\lambda)\,dv\\
&=\int_{\mathcal{C}}\oint_{\mathcal{C}'}\frac{z}{z-e^{2\pi\iota v}}G_i^{\pm}\left(e^{2\pi\iota v}\right)^{-1}\frac{\theta(v-v'+\lambda_i)}{\theta(v-v')\theta(\lambda_i)}G_i^{\pm}\left(e^{2\pi\iota v'}\right)\mathcal{X}^{\pm}\left(e^{2\pi\iota v'}\right)\,dv'\,dv
\end{aligned}
$$

Again we integrate with respect to v first, assuming $\mathcal{C}'$ is contained in $\mathcal{C}$, to get

$$
\oint_{\mathcal{C}'}\frac{z}{z-e^{2\pi\iota v'}}\mathcal{X}_i^{\pm}\left(e^{2\pi\iota v'}\right)(2\pi\iota)dv'
$$

Setting $w=e^{2\pi\iota v'}$, the last expression equals

$$
\oint_{\widetilde{\mathcal{C}'}}\frac{z}{z-w}\mathcal{X}_i^{\pm}(w)\frac{dw}{w}=\mathcal{X}_i^{\pm}(z)
$$

where $\widetilde{\mathcal{C}'}=\exp(2\pi\iota\mathcal{C}')$.

9. Classification of irreducibles II: Sufficient condition

9.1. **Classification theorem.** Consider $\mathbb{V}\in\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ an irreducible object. Let $\mu\in\mathfrak{h}^*$ be a weight of $\mathbb{V}$ such that $\mathbb{V}_{\mu+\alpha_i}=0$ for every $i\in\mathbf{I}$. Choose $\Omega\in\mathbb{V}_\mu$ a non–zero eigenvector for $\{\Phi_i(u)\}_{i\in\mathbf{I}}$. That is, there exists $\underline{\mathcal{A}}=(A_i(u))\in\mathcal{M}(\mu)$ (see Section 3.4 for definition of $\mathcal{M}(\mu)$) such that $\Phi_i(u)\Omega=A_i(u)\Omega$.

Theorem.

(1) *For each $i\in\mathbf{I}$ there exists $N_i\in\mathbb{N}$, $b_1^{(i)},\cdots,b_{N_i}^{(i)}\in\mathbb{C}$, and a constant $C_i\in\mathbb{C}^\times$ such that*

$$
A_i(u)=C_i\prod_{k=1}^{N_i}\frac{\theta(u-b_k^{(i)}+d_i\hbar)}{\theta(u-b_k^{(i)})}
$$

(in particular $\mu(\alpha_i^\vee)=N_i\in\mathbb{N}$ and hence μ is a dominant integral weight). We say that the data of such $(\mu,\mathbf{b})$ is the l–highest weight of $\mathbb{V}$, where

$$
\mathbf{b}=(\mathbf{b}_i=\{b_1^{(i)},\cdots,b_{N_i}^{(i)}\})_{i\in\mathbf{I}}
$$

(2) *Let $\mathbb{V}$ and $\mathbb{W}$ be two irreducible objects of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ with l–highest weights $(\mu,\mathbf{b})$ and $(\nu,\mathbf{c})$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if, and only if $\mu=\nu$ and for every $i\in\mathbf{I}$ there exists $\sigma_i\in\mathfrak{S}_{N_i}$ such that $b_k^{(i)}\equiv c_{\sigma_i(k)}^{(i)}$ modulo Λ_τ.*

(3) *For every $(\mu,\mathbf{b})$ such that $\mu\in\mathfrak{h}^*$ is a dominant integral weight and $\mathbf{b}=(\mathbf{b}_i)$ where each $\mathbf{b}_i$ is an unordered $\mu(\alpha_i^\vee)$–tuple of complex numbers, there exists an irreducible object $\mathbb{V}(\mu,\mathbf{b})$ of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ of l–highest weight $(\mu,\mathbf{b})$.*

Proof. (1) follows from Proposition 3.6 and Theorem 4.1.

(2): Fix a subset $\mathrm{S}\subset\mathbb{C}$ as in Section 8.1. Using Propositions 3.6, 8.3 and Theorem 8.6 we know that there exist (necessarily irreducible) representations $\mathcal{V},\mathcal{W}\in\mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(U_q(L\mathfrak{g}))$ such that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic to $\Theta(\mathcal{V})$ and $\Theta(\mathcal{W})$ respectively. If l–weights of $\mathbb{V}$ and $\mathbb{W}$ are the same as stated in (2), then $\mathcal{V}$ and $\mathcal{W}$ have the same Drinfeld polynomials and the result follows from the classification theorem for irreducible representations of quantum loop algebras (Theorem 5.4).

(3): Now let $(\mu,\mathbf{b})$ be as given above. Note that by (1) the data of l–highest weight for an irreducible object of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$ depends only on the equivalence class of $b_k^{(i)}$ modulo Λ_τ. Thus we can assume without loss of generality that $b_k^{(i)}\in\mathrm{S}$ for each $i\in\mathbf{I}$ and $1\leq k\leq N_i=\mu(\alpha_i^\vee)$. Let $\mathcal{V}$ be the irreducible representation of $U_q(L\mathfrak{g})$ of l–highest weight given by $(\mu,\{\mathcal{P}_i(w)\})$ where the Drinfeld polynomials $\mathcal{P}_i$ are given by

$$\mathcal{P}_i(w)=\prod_{k=1}^{N_i}\left(w-e^{2\pi\iota b_k^{(i)}}\right)$$

Let $\Omega\in\mathcal{V}_\mu$ be the (unique up to scalar) highest weight vector. Thus the action of the commuting elements $\{\Psi_i(z)\}$ on Ω is given by

$$\Psi_i(z)\Omega=\prod_{k=1}^{N_i}\frac{q_iz-q_i^{-1}\beta_k^{(i)}}{z-\beta_k^{(i)}}$$

where $\beta_k^{(i)}=\exp\left(2\pi\iota b_k^{(i)}\right)$.

Now we have the following calculation for the action of $\Phi_i(u)$ on Ω, using the definition (6.2) and identity (2.1).

$$\begin{aligned}\Phi_i(u)\Omega&=\prod_{k=1}^{N_i}\left(\left(\prod_{n\geq1}\frac{1-(q_i^2z/\beta_k^{(i)})p^n}{1-(z/\beta_k^{(i)})p^n}\right)\frac{q_iz-q_i^{-1}\beta_k^{(i)}}{z-\beta_k^{(i)}}\left(\prod_{n\geq1}\frac{1-(\beta_k^{(i)}/q_i^2z)p^n}{1-(\beta_k^{(i)}/q_i^2z)}\right)\right)\Omega\\&=\prod_{k=1}^{N_i}\frac{\theta(u-b_k^{(i)}+d_i\hbar)}{\theta(u-b_k^{(i)})}\Omega\end{aligned}$$

Thus it only remains to show that $\mathbb{V}$ is irreducible. Assuming the contrary, let $\mathbb{V}_1\subset\mathbb{V}$ be a non–zero proper subspace which is stable under the action of $\{\Phi_i(u),\mathfrak{X}_i^\pm(u,\lambda)\}$. Then $\mathcal{V}_1=\Xi(\mathbb{V}_1)$ is a non–zero proper subrepresentation of $\mathcal{V}$ contradicting its irreducibility. $\square$

9.2. **Schur's Lemma.**

Proposition. *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be two irreducible objects of* $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$*. Then either* $\mathbb{V}$ *and* $\mathbb{W}$ *are isomorphic, or* $\mathrm{Hom}_{\mathcal{L}_{\hbar,\tau}(\mathfrak{g})}(\mathbb{V},\mathbb{W}) = \{0\}$*.*

PROOF. By Corollary 8.4 we can assume that both $\mathbb{V}$ and $\mathbb{W}$ are in $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$. Let $\varphi(\lambda) : \mathbb{V} \to \mathbb{W}$ be a morphism. Let $\Omega_{\mathbb{V}} \in \mathbb{V}_\mu$ be the (up to a scalar) highest–weight vector of $\mathbb{V}$. For every $i \in \mathbf{I}$, let $N_i = \mu(\alpha_i^\vee)$ and let $b_1^{(i)}, \cdots, b_{N_i}^{(i)} \in \mathrm{S}$ be such that

$$\Phi_i(u)_{\mathbb{V}}\Omega_{\mathbb{V}} = \prod_{k=1}^{N_i} \frac{\theta(u - b_k^{(i)} + d_i\hbar)}{\theta(u - b_k^{(i)})}\Omega$$

Below we will just write $A_i(u)$ for the eigenvalue of $\Phi_i(u)$ on $\Omega_{\mathbb{V}}$.

Let us define $\Omega(\lambda) = \varphi(\lambda)(\Omega_{\mathbb{V}})$ in $\mathbb{W}_\mu$. According to the definition of morphisms (see Section 2.3) we obtain the following two identities for each $i \in \mathbf{I}$:

$$\Phi_i(u)_{\mathbb{W}}\Omega(\lambda) = A_i(u)\Omega(\lambda + \hbar\alpha_i) \tag{9.1}$$

$$\mathfrak{X}_i^+(u,\lambda)\Omega\left(\lambda - \frac{\hbar}{2}(\mu + \alpha_i)\right) = 0 \tag{9.2}$$

Recall that by Proposition 8.3 we know the following expression for $\mathfrak{X}_i^+(u,\lambda)$ acting on $\mathbb{W}_\mu$:

$$\mathfrak{X}_i^+(u,\lambda) = \sum_{\substack{a\in\mathrm{S}\\ n\in\mathbb{N}}} \frac{\partial_u^n}{n!}\left(\frac{\theta(u - a + \lambda_i)}{\theta(u-a)\theta(\lambda_i)}\right)\mathsf{X}^+_{i;a,n}$$

where the sum is finite and $\mathsf{X}^+_{i;a,n}$ do no depend on the dynamical variable. Now we can multiply equation (9.2) by $(u-a)^n$ and integrate over a small contour around a to get that $\mathsf{X}^+_{i;a,n}$ annihilate $\Omega(\lambda)$ for all λ. Hence, for $\lambda_0 \in \mathfrak{h}^*$ such that $\varphi(\lambda_0)$ is defined on $\mathbb{V}_\mu$, $\Omega(\lambda_0) \in \mathbb{W}_\mu$ is annihilated by all the raising operators.

By irreducibility of $\mathbb{W}$, if μ is not the highest–weight of $\mathbb{W}$, we get that $\varphi(\lambda)\Omega_{\mathbb{V}} = 0$. By Theorem 3.8, this means that $\varphi(\lambda) \equiv 0$.

Now assume that the highest–weight of $\mathbb{W}$ is also μ and let $\Omega_{\mathbb{W}} \in \mathbb{W}_\mu$ be the unique (up to scalar) highest–weight vector of $\mathbb{W}$. Thus we have a scalar function $\phi(\lambda)$ such that $\Omega(\lambda) = \phi(\lambda)\Omega_{\mathbb{W}}$. Let us write $B_i(u)$ for the eigenvalue of $\Phi_i(u)$ acting on $\Omega_{\mathbb{W}}$. Then by (9.1) we get

$$\phi(\lambda)B_i(u) = \phi(\lambda + \hbar\alpha_i)A_i(u)$$

Thus $B_i(u) = C_iA_i(u)$ for some $C_i \in \mathbb{C}^\times$. We know from Proposition 8.3 that $C_i = 1$. This proves that $\mathbb{V}$ and $\mathbb{W}$ have the same l–highest weight, and hence so must $\mathcal{V} = \Xi(\mathbb{V})$ and $\mathcal{W} = \Xi(\mathbb{W})$. Let $\psi : \mathcal{V} \to \mathcal{W}$ be the isomorphism in $\mathcal{O}^{\mathrm{S}}_{\mathrm{int}}(U_q(L\mathfrak{g}))$. Then ψ is also an isomorphism between $\mathbb{V} = \Theta(\mathcal{V})$ and $\mathbb{W} = \Theta(\mathcal{W})$ and we are done.

□

9.3.

Corollary. *Let $\mathbb{V}$ and $\mathbb{W}$ be two objects of $\mathcal{L}_{\hbar,\tau}(\mathfrak{g})$. Assume $\mathbb{W}$ is irreducible and $\mathbb{V} \neq \{0\}$. Let there be an injective morphism $\varphi(\lambda) : \mathbb{V} \to \mathbb{W}$. Then $\varphi(\lambda)$ is an isomorphism.*

Proof. Again using Corollary 8.4 we can assume that $\mathbb{V}$ and $\mathbb{W}$ are in $\mathcal{L}^{\mathrm{S}}_{\hbar,\tau}(\mathfrak{g})$. Let $\mathbb{V}_1 \subset \mathbb{V}$ be a minimal (with respect to inclusion) non–zero subspace which is stable under the action of $\{\Phi_i(u), \mathfrak{X}_i^{\pm}(u,\lambda)\}$. Using Proposition 9.2 we know that either $\varphi(\lambda)$ restricted to $\mathbb{V}_1$ is zero; or $\mathbb{V}_1$ and $\mathbb{W}$ are isomorphic, via an isomorphism which is independent of λ. In the former case we contradict the injectivity of $\varphi(\lambda)$. In the latter case, let us use the isomorphism between $\mathbb{V}_1$ and $\mathbb{W}$ to identify the two.

This brings us to the situation where we have $j : \mathbb{W} \subset \mathbb{V}$ as a subspace (stable under the action of $\{\Phi_i(u), \mathfrak{X}_i^{\pm}(u,\lambda)\}$) and $\varphi(\lambda) : \mathbb{V} \to \mathbb{W}$ such that $\varphi(\lambda)$ restricted to $\mathbb{W}$ is identity. In other words, $\psi(\lambda) = j \circ \varphi(\lambda)$ is injective (being a composition of two injective morphisms) and a projection (that is, $\psi \circ \psi = \psi$). Now it is a general fact (also not so hard to prove) that an injective projection has to be identity, which finishes the proof of the corollary.

□

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